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Bruno FORNET

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Problèmes Hyperboliques à Coefficients Discontinus et Pénalisation de Problèmes Hyperboliques

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RAPPORTEURS

Mme Sylvie Benzoni-Gavage
M. Philippe G. LeFloch

JURY

Mme Sylvie Benzoni-Gavage	Université Lyon 1	*Rapporteur*
M. François Bouchut	ENS Ulm	*Président du jury*
M. Thierry Gallouët	Université de Provence	*Examinateur*
M. Olivier Guès	Université de Provence	*Directeur de thèse*
M. Frédéric Rousset	Université de Nice Sophia-Antipolis	*Examinateur*

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Résumé:

Les résultats contenus dans ce mémoire de Thèse concernent des problèmes hyperboliques du premier ordre et se divisent en deux parties.

La première partie du mémoire porte sur des problèmes de Cauchy linéaires dont les coefficients sont discontinus au travers d'une interface fixée, supposée non-caractéristique. De tels problèmes n'ont en général pas de sens classique. Ce type de problématique est bien connu et apparaît suite à la modélisation de certains phénomènes physiques. Nous choisissons une approche à viscosité évanescente pour aborder la question. L'existence, l'unicité et la stabilité de la solution à petite viscosité sont établies dans divers cadres incluant des problèmes scalaires et des systèmes pour des opérateurs hyperboliques formulés sous forme conservative ou non-conservative. La nature de l'interface est analysée en termes de modes compressifs, expansifs et traversants; chaque type de modes s'accompagnant d'un comportement qualitatif différent de la solution.

Dans la deuxième partie du mémoire, la question abordée est celle de l'approximation de solutions de problèmes aux limites hyperboliques au moyen de méthodes de pénalisation de domaine. Le principe des méthodes de pénalisation de domaine est de remplacer un problème avec conditions aux limites par un problème sans condition aux limites, défini sur un domaine plus large, appelé domaine fictif. Deux types de problèmes mixtes hyperboliques sont envisagés. Ces problèmes sont respectivement bien posés au sens de Friedrichs et bien posés au sens de Kreiss. Pour les problèmes bien posés au sens de Friedrichs, deux méthodes de pénalisation sont proposées. L'une d'entre elles a l'avantage de permettre l'approximation de la solution du problème aux limites hyperboliques considéré, sans formation de couches limites. Nous montrons, en particulier, que si le problème considéré est posé sur un ouvert borné régulier, nous pouvons choisir comme domaine fictif un ouvert borné parallélépipédique. Pour les problèmes Kreiss-symétrisables, dans un cadre plus restrictif, deux méthodes de pénalisation microlocales sont données; chacune permet l'approximation de la solution du problème mixte hyperbolique considéré, sans phénomène de couches limites.

Mots-clefs: problèmes hyperboliques à coefficients discontinus, problèmes mixtes hyperboliques, problèmes de perturbations singulières, couches limites, méthodes de pénalisation de domaine, conditions spectrales de stabilité.

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Chapter 1

Introduction

Cette Thèse s'articule autour de deux thèmes:

1/ L'étude de problèmes linéaires hyperboliques à coefficients discontinus.

2/ L'approximation de solutions de problèmes aux limites hyperboliques.

La première partie de cette Thèse est consacrée à des problèmes hyperboliques linéaires dont les coefficients sont discontinus. D'un point de vue physique, de tels problèmes apparaissent, par exemple, lorsque l'on considère la propagation d'une onde, ou d'un fluide, sur un domaine constitué de deux matériaux aux propriétés physiques différentes. En général, la transition entre les deux milieux se manifeste par une discontinuité des coefficients au niveau de l'EDP modélisant le problème. D'un point de vue mathématique, ces problèmes se révèlent être très semblables à des linéarisés d'ondes de chocs. Ce type de problématique est néanmoins à distinguer des modélisations de chocs, dans lesquelles la singularité de la solution apparaît le long d'un front inconnu a priori. Dans mon cas, je supposerai que les deux milieux sont situés de part et d'autre d'une hypersurface connue : l'interface. L'hétérogénéité du milieu impose alors la discontinuité du coefficient le long de cette interface. On rencontre de tels problèmes, par exemple, lorsque l'on fait l'étude de l'écoulement d'un fluide dans un milieu poreux hétérogène ([Bac05]). Dans [LP60], les auteurs traitent d'un problème issu de l'acoustique, dont l'équation est de la forme:

$$\partial_t u + A(x)\partial_x u = 0,$$

où A est une matrice de $\mathcal{M}_2(\mathbb{R})$, discontinue de part et d'autre d'une interface d'équation $\{x = \alpha\}$. Même s'il paraît naturel que de telles équations existent, l'analyse mathématique du problème n'est pas évidente. Ainsi, l'étude de telles équations constitue-t-elle une problématique très intéressante.
Si d'une part on s'attend à des solutions discontinues, d'autre part cette discontinuité crée des difficultés. En effet, si le coefficient A et la solution u sont tous deux discontinus au travers de l'hypersurface d'équation $\{x = \alpha\}$, le sens à donner au produit non-conservatif $A(x)\partial_x u$ n'est plus du tout évident. Plaçons-nous un intant dans le cas scalaire 1-D pour simplifier les choses. Je m'intéresserai à la fois à des problèmes dans leur formulation non-conservative (l'opérateur s'écrit $\partial_t + a(t,x)\partial_x$) et conservative (l'opérateur s'écrit $\partial_t u + \partial_x(a(t,x).))$. Comme le souligne par exemple P. G. LeFloch dans [LeF90], les comportements observés dans ces deux cas sont différents. En particulier, une difficulté spécifique apparaît dans le cadre non-conservatif : celle du sens à donner au produit non-conservatif. Il est à noter que le premier article d'analyse sur les produits non-conservatifs est [DMLM95] par G. Dal Maso, P. G. LeFloch et F. Murat. D'autres définitions rigoureuses de tels produits existent. On pourra par exemple se référer à la comparaison de ces définitions effectuée par P. G. LeFloch et A. E. Tzavaras dans [LT99]. Cela aboutit en particulier à des résultats d'existence et de stabilité ([LeF90]) pour des équations scalaires non-conservatives à coefficients discontinus. Les résultats de P. G. LeFloch dans [LeF90], en ce qui concerne les problèmes non-conservatifs, ont ensuite été généralisés à des systèmes 1-D par G. Crasta et P. G. LeFloch dans [CL02]. Des résultats analogues pour des problèmes conservatifs à coefficients discontinus ont été prouvés dans [LeF90] (cas scalaire) par P. G. LeFloch et dans [HL96] (systèmes 1-D) par J. Hu et P. G. LeFloch. Pour ma part, le problème considéré est linéaire (comme dans [LeF90] et [CL02]) et je suppose que la discontinuité du coefficient a lieu le long d'une hypersurface non-caractéristique. Une approche à viscosité évanescente se révèle alors être adaptée à l'étude du problème.

La première partie de la Thèse traite donc du thème des problèmes hyperboliques à coefficients discontinus et se divise en trois chapitres, numérotés de 2 à 4, dont je vais maintenant résumer le contenu.

Dans le chapitre 2 de cette Thèse, je résous deux questions ouvertes pour des problèmes scalaires conservatifs monodimensionnels. Le problème considéré est le problème de Cauchy hyperbolique à coefficients discontinus suivant:

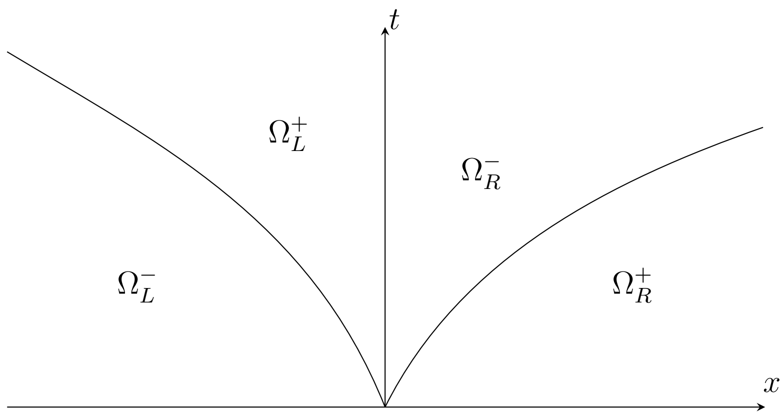
$$\begin{cases} \partial_t u + \partial_x (a(t,x)u) = f, & (t,x) \in (0,T) \times \mathbb{R}, \\ u|_{t=0} = h & , \end{cases}$$

où $T > 0$ est fixé arbitrairement, une fois pour toutes. Le coefficient a est supposé être régulier par morceaux, discontinu seulement au travers de l'hypersurface d'équation $\{x = 0\}$. f et h sont supposées C^∞ et bornées, ainsi que toutes leurs dérivées. Je suppose également que $a(t, 0^+)$ et $a(t, 0^-)$ gardent le même signe pour tout $t \in (0, T)$. Qualitativement, l'effet de la discontinuité du coefficient, sur la solution u, dépend de la configuration des caractéristiques passant par la zone de discontinuité, ici $\{x = 0\}$. Trois cas se dégagent tout naturellement :

1/ Si $a(t, 0^+) < 0$ et $a(t, 0^-) > 0$, on appelle cela le cas rentrant ou compressif. Dans ce cas, l'information contenue par la donnée de Cauchy se propage le long des caractéristiques jusqu'au bord, et ce, de chaque côté de celui-ci. Je montre la formation d'une masse de Dirac concentrée sur $\{x = 0\}$, au passage à la limite visqueuse. Il est à noter que la solution, ainsi obtenue, coïncide avec la notion de solution généralisée introduite par F. Poupaud et M. Rascle ([PR97]), en utilisant la notion de caractéristiques généralisées au sens de Filippov. J'obtiens en outre des résultats de stabilité (estimations d'énergie sur le problème visqueux).

2/ Si $a(t, 0^+) > 0$ et $a(t, 0^-) < 0$, on appelle cela le cas sortant ou expansif. Dans ce cas, on voit facilement qu'il existe une infinité de solutions faibles au problème ; la question est donc celle du choix d'une solution. Par exemple, il suffit d'imposer arbitrairement la trace $u|_{x=0}$ pour construire une solution du problème de Cauchy.
Pour les restrictions du problème de part et d'autre de l'interface, deux types de caractéristiques sont à distinguer : celles véhiculant l'information donnée par la condition de Cauchy et celles transportant l'information donnée par la trace $u|_{x=0}$. Ces deux types de caractéristiques sont séparés par les deux courbes caractéristiques issues du point $(t = 0, x = 0)$.

Figure 1



Les courbes caractéristiques issues de l'origine découpent l'espace-temps en quatre sous-domaines : Ω_L^-, Ω_L^+, Ω_R^-, Ω_R^+, sur lesquels je montrerai que la solution généralisée naturelle du problème ne présente pas de singularité.

Je prouve qu'une approche à viscosité évanescente permet de sélectionner une unique solution à ce problème. A ma connaissance, l'obtention d'un résultat d'unicité pour le cas linéaire expansif semble nouveau. De plus, comme dans le cas précédent, le résultat est accompagné d'un théorème de stabilité avec estimations d'énergie L^2. La solution à petite viscosité sélectionnée vérifie $u|_{x=0} = 0$. En particulier, ce résultat est indépendant de la viscosité choisie. Etant donné qu'il n'existe aucune corrélation entre le choix de la donnée de Cauchy et la trace obtenue à la limite visqueuse, des singularités de contact (sauts de la solution) apparaissent le long des deux courbes caractéristiques passant par le point $(t = 0, x = 0)$.

Le cas 3/ est le plus simple et n'est pas traité dans ce chapitre:

3/ Si $a(t, 0^+)$ et $a(t, 0^-)$ ont le même signe, on appelle cela le cas traversant. Une approche à petite viscosité montre qu'il suffit de rajouter une condition de raccord des flux à l'interface (continuité de $a(t, x)u(t, x)$) pour obtenir un problème bien posé et stable par petite perturbation visqueuse.

Il est à noter qu'au cours de ce chapitre, les preuves de stabilité (estimations d'énergie L^2) sont effectuées par intégration par parties. De plus, le cas expansif se prête mieux à ces estimations que le cas compressif, pour lequel j'ai dû procéder par intégration de l'équation.

Dans le chapitre 3 de cette Thèse, je m'intéresse à des systèmes hyperboliques linéaires à coefficients C^∞ par morceaux en plusieurs dimensions d'espace. Comme précédemment, la discontinuité des coefficients est localisée sur une unique hypersurface fixe. Par souci de simplicité, je suppose qu'une équation de cette hypersurface est donnée par $\{x_d = 0\}$. L'opérateur hyperbolique considéré est de la forme :

$$\partial_t + \sum_{j=1}^{d} A_j(t, y, x)\partial_j \quad ,$$

avec $\partial_j := \partial_{x_j}$, la variable y regroupant les variables d'espaces tangentielles au bord: $(x_1, \ldots, x_{d-1})$ et x désignant la variable normale au bord: $x := x_d$. De plus, les matrices A_j sont des matrices de $\mathcal{M}_N(\mathbb{R})$. Un point important est que cette hypersurface est supposée non-caractéristique, ce qui signifie que le coefficient de dérivée normale, A_d, est inversible sur l'interface.

Il est à noter que, contrairement au problème traité dans le chapitre précédent, l'opérateur considéré se présente sous une forme non-conservative. En particulier, en ce qui concerne la solution généralisée naturelle du problème, les seules singularités observées ici sont des discontinuités et aucune masse de Dirac ne se forme.

Dans ce travail, je m'inspire fortement des hypothèses dégagées lors de l'étude des ondes de choc par petite perturbation visqueuse. Ces dernières ont fait récemment l'objet d'une série d'articles par O. Guès, G. Métivier, M. Williams et K. Zumbrun (voir par exemple [GMWZ05]). Sous des hypothèses convenables de structure et de stabilité, je démontre par ailleurs la convergence de la solution du problème visqueux vers une solution limite, ce qui constitue le résultat principal de ce chapitre. La solution limite satisfait le problème hyperbolique de part et d'autre de l'interface et, de plus, des conditions de transmission sur l'interface qui sont l'analogue des conditions de Rankine-Hugoniot dans le cas des chocs.

Ce chapitre comporte un autre résultat intéressant. Les hypothèses de structure et de stabilité écartent de manière naturelle la présence de modes expansifs comme ceux étudiés lors du chapitre précédent. Néanmoins, il m'a paru intéressant d'aborder la stabilité du cas expansif en utilisant la notion de fonction d' Evans et de symétriseur, d'autant plus que, dans le cadre non-conservatif, la méthode d'estimation par intégration par parties ne fonctionne pas aussi bien. C'est ce que j'ai fait dans la dernière partie de ce chapitre, en obtenant un résultat similaire à celui du chapitre 2, cette fois dans le cas non-conservatif.

Il est à noter que la preuve des estimations d'énergie qui établissent la stabilité L^2 du problème visqueux est ici basée sur des estimations par Symétriseur de type Kreiss effectuées sur une version Laplace-Fourier du problème. Cela reste vrai aussi bien pour mon étude des systèmes en plusieurs dimensions d'espace, que pour le cas scalaire expansif que je traite à part, dans la section 3.3.

Dans Le chapitre 4 **de cette Thèse,** je cherche à étendre le cadre du chapitre 3 en y incluant la présence de modes expansifs. La définition même d'un mode expansif n'est pas tout à fait triviale. Pour simplifier, je me suis restreint au cadre des systèmes 1-D, avec des coefficients constants par morceaux. Je considère ainsi le problème de Cauchy associé à l'opérateur hyperbolique $\partial_t + A(x)\partial_x$, avec $A(x) = A^+\mathbf{1}_{x>0} + A^-\mathbf{1}_{x<0}$. L'inconnue du problème $u(t,x)$ appartient à $\mathbb{R}^N$ et $A^\pm$ est une matrice inversible de $\mathcal{M}_N(\mathbb{R})$. Le cadre d'étude est ici significativement élargi. Ma nouvelle étude englobe, en particulier, le cas purement expansif pour lequel A^+ a toutes ses valeurs propres > 0 et A^- a toutes ses valeurs propres < 0. L'ingrédient clef de ma démonstration réside dans l'identification d'un sous-espace de $\mathbb{R}^N$ sur lequel les comportements expansifs se polarisent. Pour les sytèmes traités dans le chapitre 3 de la Thèse, ce sous-espace se résumait systématiquement au $\{0_{\mathbb{R}^N}\}$, de par mes hypothèses.

Etant donné la quantité et la complexité des hypothèses, il m'a paru intéressant d'exhiber des exemples d'application de mes résultats. Je m'y suis employé pour des systèmes 2×2 non-triviaux.

Je vais maintenant donner un court résumé de la partie pénalisation de cette Thèse. Cette partie se divise en deux travaux exposés respectivement dans les chapitres 5 et 6. Comme mentionné précédemment, le but est d'approcher la solution de certains problèmes mixtes hyperboliques. Il existe deux grandes classes de problèmes mixtes hyperboliques bien posés : ceux bien posés au sens de Friedrichs et ceux bien posés au sens de Kreiss. Ces deux classes de problèmes séparent respectivement les problèmes symétrisables dans un sens classique de ceux symétrisables au sens pseudo-différentiel. Beaucoup de problèmes issus de la physique s'inscrivent dans l'une de ces deux classes de problèmes bien posés. Chaque chapitre traitera de l'approximation d'un type différent de problèmes mixtes hyperboliques.

Le chapitre 5 **de cette Thèse** est dédié à l'approximation de solutions de certains problèmes mixtes hyperboliques semi-linéaires, à bord caractéristique (à multiplicité constante), ou non-caractéristique, bien posés au sens de Friedrichs. Ce chapitre contient un papier co-écrit avec O. Guès. Pour être plus précis, les méthodes de pénalisation de domaine, proposées dans ce papier, permettent d'approximer la solution de problèmes mixtes hyperboliques semi-linéaires, dont la condition au bord est strictement maximale dissipative.
Concrètement, les problèmes considérés sont:

$$\begin{cases} Lu = F(t,x,u) \text{ sur }]0,T[\times\Omega, \\ u_{|]0,T[\times\partial\Omega} \in \mathcal{N}, \\ u_{t=0} = 0, \end{cases} \tag{1.0.1}$$

où Ω est un ouvert de $\mathbb{R}^d$ à bord C^∞, L est un système matriciel symétrique hyperbolique du premier ordre, $\mathcal{N}$ est un fibré vectoriel C^∞ sur $\mathbb{R}\times\partial\Omega$ définissant les conditions au bord, et F est une application C^∞ qui peut être non-linéaire.

Nous montrons que nous pouvons adopter deux méthodes différentes de pénalisation de domaine pour arriver à nos fins. La première méthode que nous proposons est une extension d'un résultat de J. Rauch ([Rau79]) et d'un résultat de C. Bardos et J. Rauch ([BR82]) au cas non-linéaire. Notre preuve du résultat est originale et s'accompagne d'une analyse asymptotique de la convergence, à tout ordre. Nous montrons ainsi que, pour cette méthode de pénalisation, des couches limites se forment dans un voisinage du bord de Ω, localisé à l'extérieur de celui-ci.

Suite à cela, nous donnons une nouvelle méthode de pénalisation de domaine, qui, elle, s'effectue sans formation de couches limites, à tout ordre. Cela dénote, au regard de la première méthode proposée, d'une amélioration de la qualité d'approximation.

Nos résultats ont été affinés dans la préoccupation de futures applications numériques. Par exemple, nous avons montré que le domaine fictif pénalisé peut être un ouvert assez quelconque contenant Ω.

Dans le chapitre 6 de cette Thèse, je m'intéresse à l'approximation de problèmes mixtes hyperboliques à coefficients constants, posés sur un demi-espace, de la forme :

$$\begin{cases} \partial_t u + \displaystyle\sum_{j=1}^{d} A_j \partial_j u = f, \quad \{x > 0\}, \\ \Gamma u|_{x=0} = \Gamma g, \\ u|_{t<0} = 0 \quad . \end{cases}$$

Les problèmes auxquels je m'intéresse sont à bord (ici $\{x = 0\}$) non-caractéristique et satisfont une condition de Lopatinski uniforme. Il est à noter que les problèmes ainsi considérés sont Kreiss-Symétrisables.

Je fournis deux méthodes différentes d'approximation de tels problèmes. Il s'agit là, probablement, du premier résultat obtenu concernant l'approximation de problèmes mixtes hyperboliques Kreiss-symétrisables par des méthodes de pénalisation de domaine.

Je propose deux méthodes différentes de pénalisation de domaine. Ces deux méthodes n'engendrent pas de couches limites, à tout ordre.

La première méthode nécessite la construction d'un symétriseur de Kreiss. Il s'agit là d'un objet microlocal.

La deuxième méthode proposée paraît plus simple, dans une perspective numérique, car elle n'utilise que des projecteurs assez naturels, et ne nécessite pas la construction d'un symétriseur de Kreiss.

Les deux méthodes exposées dans ce chapitre diffèrent en profondeur l'une de l'autre. Aussi, n'ai-je pas pu donner une preuve de stabilité commune aux deux méthodes proposées, contrairement à ce qui avait été fait au chapitre 5.

On va maintenant détailler davantage le contenu de chacun des chapitres de cette Thèse.

1.1 Problèmes linéaires hyperboliques conservatifs à coefficients discontinus : le cas scalaire 1-D (Chapitre 2).

Je m'intéresse ici à des problèmes linéaires hyperboliques de la forme :

$$\begin{cases} \partial_t u + \partial_x(a(t,x)u) = f, & x \in \mathbb{R}, \\ u|_{t=0} = h \quad , \end{cases} \tag{1.1.1}$$

dans le cas où le coefficient a est discontinu au travers de $\{x = 0\}$, et régulier de part et d'autre.

En fait, l'étude d'une telle équation nécessite l'introduction d'une nouvelle notion de solutions. Les problèmes hyperboliques à coefficients peu réguliers ont déjà fait l'objet de plusieurs travaux : R.J. Diperna et P.-L. Lions ([DL89]) ont défini une notion de solutions renormalisées pour ce genre de problèmes; P. G. LeFloch a résolu un problème voisin dans [LeF90]; F. Bouchut et F. James ([BJ98]) ont introduit une notion de solution faible adaptée, s'articulant autour de l'étude parallèle du problème pris dans sa version conservative et sa version non-conservative. Ces résultats ont été étendus par F. Bouchut, F. James et S. Mancini dans [BJM05]. F. Poupaud et M. Rascle ([PR97]) ont proposé une notion de solution basée sur les caractéristiques généralisées au sens de Filippov.

Pour ma part, je me focalise sur le cas où le coefficient a est une fonction régulière par morceaux de part et d'autre de l'hypersurface d'équation $\{x = 0\}$. L'approche choisie pour aborder le problème est une approche à viscosité évanescente. Soit $T > 0$ fixé arbitrairement; le problème sera étudié sur la fenêtre temporelle $(0, T)$.
Je choisis ensuite les hypothèses de manière à me concentrer sur l'effet provoqué par la discontinuité du coefficient. A cette fin, la fonction $(t, x) \to a(t, x)$ est supposée C^∞ de part et d'autre de la droite $\{x = 0\}$, plus précisément je prendrai

$$a \in C_b^\infty((0, T) \times \mathbb{R}^*),$$

où C_b^∞ désigne l'espace des fonctions infiniment différentiables, bornées, ainsi que chacune de leurs dérivées.
Il est à noter que, dans la définition, $a|_{x=0}$ n'est pas définie. En fait, pour mon approche, la valeur de a en $\{x = 0\}$ n'a pas d'importance. De plus, je suppose que le terme source f appartient à l'ensemble des fonctions C^∞ à support compact $C_0^\infty((0, T) \times \mathbb{R})$ et que h appartient à l'ensemble des fonctions tests $C_0^\infty(\mathbb{R})$.

Je considère maintenant le problème, parabolique à $\varepsilon > 0$ fixé, obtenu par la perturbation visqueuse suivante du problème de Cauchy hyperbolique (1.1.1):

$$\begin{cases} \partial_t u^\varepsilon + \partial_x \left(a(t, x) u^\varepsilon\right) - \varepsilon \partial_x^2 u^\varepsilon = f, \qquad x \in \mathbb{R}, \\ u^\varepsilon|_{t=0} = h \quad . \end{cases} \tag{1.1.2}$$

Or, les travaux précédents suggèrent que, dans les cas dits rentrants, la solution généralisée naturelle du problème comporte une masse de Dirac en $\{x = 0\}$, de sorte que le produit au n'a pas de sens classique. Même si le problème (1.1.1) n'a pas toujours de solution naturelle au sens des distributions, il n'en est pas de même du problème (1.1.2) pris à $\varepsilon > 0$ fixé ([Ike71]). Ceci est dû au caractère parabolique du problème considéré ou encore à l'effet régularisant du Laplacien.

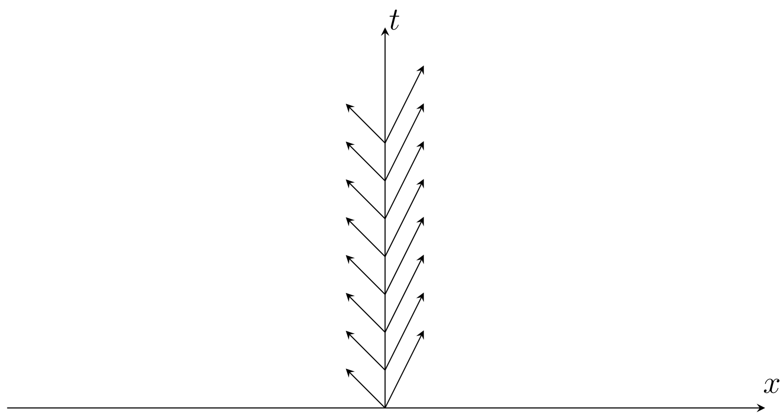
Ainsi, dans le cas présent, malgré la discontinuité du coefficient a en $\{x = 0\}$, à $\varepsilon > 0$ fixé, la solution u^ε de (1.1.2) appartient à $C^0((0,T) \times \mathbb{R})$. Plus précisément, si je note $u^{\varepsilon+}$ et $u^{\varepsilon-}$ les restrictions de u^ε respectivement à $\{x > 0\}$ et $\{x < 0\}$, u^ε satisfait les conditions de transmission à l'interface :

$$\begin{cases} u^{\varepsilon+}|_{x=0^+} - u^{\varepsilon-}|_{x=0^-} = 0, \\ \left(au^{\varepsilon+} - \varepsilon\partial_x u^{\varepsilon+}\right)|_{x=0^+} - \left(au^{\varepsilon-} - \varepsilon\partial_x u^{\varepsilon-}\right)|_{x=0^-} = 0 \end{cases} .$$

Ce chapitre comporte deux résultats principaux : l'un portant sur le cas expansif ($a(t,0^-) < 0$, $a(t,0^+) > 0$) et l'autre concernant le cas compressif ($a(t,0^-) > 0$, $a(t,0^+) < 0$).

Figure 2

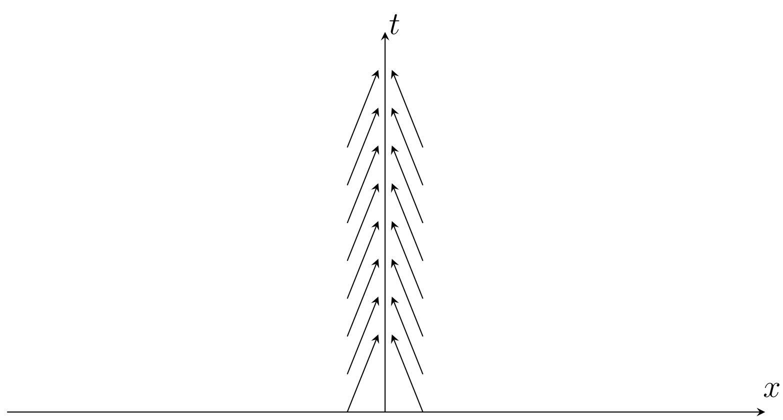
Cas Expansif



Représentation du champ de vecteurs $\partial_t + a(t,x)\partial_x$ au voisinage de l'interface pour un cas expansif

Figure 3

Cas Compressif



Représentation du champ de vecteurs $\partial_t + a(t,x)\partial_x$ au voisinage de l'interface pour un cas compressif

1.1.1 Traitement du cas expansif.

Mon résultat principal montre que, dans le cas expansif, la solution u^ε de (1.1.2) converge, quand $\varepsilon \to 0^+$ vers $\underline{u}$, définie par: $\underline{u}|_{x=0} = 0$, $\underline{u}|_{x>0} = u_R$ et $\underline{u}|_{x<0} = u_L$ où (u_L, u_R) est l'unique solution du problème hyperbolique suivant:

$$\begin{cases} \partial_t u_R + \partial_x(a_R u_R) = f_R, & \{x > 0\}, \\ \partial_t u_L + \partial_x(a_L u_L) = f_L, & \{x < 0\}, \\ u_R|_{x=0} = u_L|_{x=0} = 0, \quad \forall t \in (0,T), \\ u_R|_{t=0} = h_R, u_L|_{t=0} = h_L \quad , \end{cases}$$

où les indices "L" et "R" servent à indiquer les restrictions de la fonction concernée respectivement à $\{x < 0\}$ et $\{x > 0\}$. La fonction u ainsi définie n'est pas dans $C^0((0,T) : H^1(\mathbb{R}))$. Cela est tout à fait normal puisque qu'aucune condition de compatibilité n'est exigée sur la donnée de Cauchy h.

Enonçons maintenant mon résultat, qui est le Théorème 2.2.3.

Théorème 1.1.1. *Il existe $C > 0$ tel que, pour tout $0 < \varepsilon < 1$, on ait*

$$\|u^\varepsilon - \underline{u}\|_{L^\infty([0,T]:L^2(\mathbb{R}))} \leq C\varepsilon^{\frac{1}{4}},$$

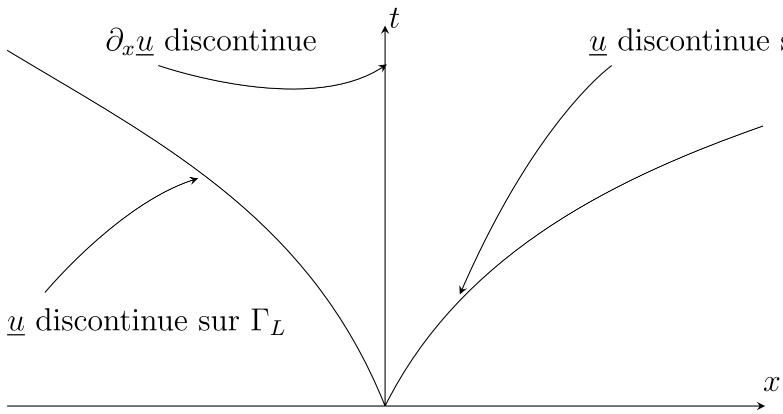
où u^ε est la solution de (1.1.2).

Ce résultat montre que l'approche visqueuse proposée parvient à sélectionner une unique solution, alors même que le problème considéré initialement possédait une infinité de solutions faibles. Il est à remarquer que ce résultat reste valable pour une viscosité de la forme: $-\varepsilon g(t,x)\partial_x^2$, où g désigne, par exemple, une fonction C^∞, uniformément définie positive et constante en dehors d'un compact.

D'après ce Théorème, la vitesse de convergence observée est en $\mathcal{O}(\varepsilon^{\frac{1}{4}})$. Cela est dû à l'apparition de couches limites caractéristiques le long des deux courbes caractéristiques issues de $(t,x) = (0,0)$, Γ_L et Γ_R.

FIGURE 4

Singularités de la solution à petite viscosité, dans le cas scalaire expansif conservatif



Les courbes caractéristiques situées au-dessus de Γ_L dans le demi-espace $\{x < 0\}$ et les courbes caractéristiques situées au-dessus de Γ_R dans le demi-espace $\{x > 0\}$ sont issues de $\{x = 0\}$ et véhiculent donc l'information donnée par la trace $u|_{x=0}$. Les courbes caractéristiques situées au-dessous de Γ_L dans le demi-espace $\{x < 0\}$ et les courbes caractéristiques situées au-dessous de Γ_R dans le demi-espace $\{x > 0\}$ sont issues de $\{t = 0\}$, et relaient donc l'information fournie par la donnée de Cauchy. Les discontinuités de contact de la solution le long de Γ_L et Γ_R proviennent de cette disparité.

Pour tout $\varepsilon > 0$ fixé, u^ε est continue le long de Γ_L et Γ_R. La transition de $u^\varepsilon = u^\varepsilon_R \mathbf{1}_{x>0} + u^\varepsilon_L \mathbf{1}_{x<0}$ vers u s'accompagne donc d'une formation de couches limites qui peut être décrite, à tout ordre (comme je l'ai montré lors de la construction de la solution approchée du problème visqueux, dans la première partie de la preuve du Théorème 2.2.3, située dans la section 2.2) par des ansatz de la forme suivante :

$$u^\varepsilon_R(t,x) \sim_{\varepsilon\to 0^+} \sum_{n\geq 0} \underline{\mathbf{U}}_{R,n}(t,x)\varepsilon^{\frac{n}{2}}$$

$$+\sum_{n\geq 0}\left(\mathbf{U}^c_{R,n,+}\left(t, \frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right)\mathbf{1}_{(t,x)\in\Omega^+_R} + \mathbf{U}^c_{R,n,-}\left(t, \frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right)\mathbf{1}_{(t,x)\in\Omega^-_R}\right)\varepsilon^{\frac{n}{2}},$$

et

$$u_L^\varepsilon(t,x) \sim_{\varepsilon\to 0^+} \sum_{n\geq 0} \underline{\mathbf{U}}_{L,n}(t,x)\varepsilon^{\frac{n}{2}}$$

$$+\sum_{n\geq 0}\left(\mathbf{U}^c_{L,n,+}\left(t,\frac{\varphi_L(t,x)}{\sqrt{\varepsilon}}\right)\mathbf{1}_{(t,x)\in\Omega_L^+} + \mathbf{U}^c_{L,n,-}\left(t,\frac{\varphi_L(t,x)}{\sqrt{\varepsilon}}\right)\mathbf{1}_{(t,x)\in\Omega_L^-}\right)\varepsilon^{\frac{n}{2}},$$

où les termes avec un exposant "c" (comme caractéristique) servent à décrire les couches limites caractéristiques se formant, et décroissent exponentiellement vite par rapport à leur deuxième variable (variable rapide). Introduisons maintenant quelques éléments géométriques nécessaires à la compréhension de cet ansatz. Tout d'abord, $\Omega_L^\pm$ et $\Omega_R^\pm$ se définissent comme cela est illustré dans la figure 1. La courbe caractéristique séparant Ω_R^+ de Ω_R^- sera notée Γ_R. De même, je noterai Γ_L la courbe caractéristique séparant Ω_L^+ de Ω_L^-. Des équations de Γ_R et Γ_L sont données par :

$$\Gamma_R = \{(t,x)\in\Omega_R : \varphi_R(t,x)=0\},$$

$$\Gamma_L = \{(t,x)\in\Omega_L : \varphi_L(t,x)=0\}.$$

En introduisant $\widetilde{a}_R$ qui est une extension C^∞ arbitraire de $a_R := a|_{x>0}$ au demi-espace $\{x<0\}$, la fonction φ_R se définit comme étant la solution de:

$$\begin{cases} (\partial_t + \widetilde{a}_R(t,x)\partial_x)\varphi_R = 0, & (t,x)\in(0,T)\times\mathbb{R}, \\ \varphi_R|_{t=0} = x \quad , \end{cases}$$

et φ_L se définit symétriquement.

1.1.2 Traitement du cas compressif.

Je reprends le même type d'analyse que précédemment. Cette fois, l'analyse asymptotique montre l'apparition de couches limites de forte amplitude se formant sur $\{x=0\}$. De même que précédemment, je donne un ansatz qui décrit avec précision la formation de couches limites, à tout ordre :

$$u^\varepsilon(t,x) \sim_{\varepsilon\to 0^+} \varepsilon^{-1}\mathbf{U}^*_{-1}\left(t,\frac{x}{\varepsilon}\right) + \sum_{n\geq 0}\left(\underline{\mathbf{U}}_n(t,x) + \mathbf{U}^*_n\left(t,\frac{x}{\varepsilon}\right)\right)\varepsilon^n,$$

où les fonctions avec l'exposant "*" décroissent exponentiellement vite en leur deuxième variable (variable rapide). Le calcul des profils $\underline{\mathbf{U}}_n$ et $\mathbf{U}_n^*$ est donné à la suite de la preuve de la Proposition 2.3.1. Le premier terme de ce développement, d'ordre de grandeur :

$$\varepsilon^{-1} e^{-\frac{x}{\varepsilon}},$$

converge vers une masse de Dirac localisée en $\{x = 0\}$.

Je prouve ainsi que, lorsque ε tend vers zéro, u^ε converge au sens des distributions vers une unique solution $\underline{u}$, qui est une mesure de la forme:

$$\underline{u}(t, .) = C(t)\ \delta_{x=0} + u_0(t, .),$$

où u_0 appartient à $L^2((0,T) \times \mathbb{R})$ et C est continue sur $(0, T)$. Ce résultat de convergence, incluant la définition de $C(t)$, est donné dans le Corollaire 2.3.3.

Il est à noter que u_0 est donnée par: $u_0 := u_R \mathbf{1}_{x>0} + u_L \mathbf{1}_{x<0}$, où u_R et u_L vérifient respectivement les problèmes hyperboliques bien posés suivants :

$$\begin{cases} \partial_t u_R + \partial_x(a_R u_R) = f_R, & \{x > 0\}, \\ u_R|_{t=0} = h_R, \end{cases}$$

$$\begin{cases} \partial_t u_L + \partial_x(a_L u_L) = f_L, & \{x < 0\}, \\ u_L|_{t=0} = h_L \quad . \end{cases}$$

On a alors :

Théorème 1.1.2. *Il existe* $C > 0$ *tel que pour tout* $0 < \varepsilon < 1$, *on ait :*

$$\|u^\varepsilon|_{x>0} - u_R\|_{L^2((0,T)\times\mathbb{R}_+^*)} \leq C\varepsilon,$$

$$\|u^\varepsilon|_{x<0} - u_L\|_{L^2((0,T)\times\mathbb{R}_-^*)} \leq C\varepsilon.$$

1.2 Systèmes linéaires hyperboliques à coefficients discontinus (Chapitre 3).

Comme je l'ai déjà mentionné auparavant, ce chapitre contient deux résultats. Commençons par mon résultat portant sur des systèmes hyperboliques, C^∞ par morceaux, en plusieurs dimensions d'espace.

1.2.1 Systèmes linéaires hyperboliques à coefficients discontinus sans modes expansifs.

Ma préoccupation première a été de donner un sens au problème suivant:

$$\begin{cases} \mathcal{H}u = f, & (t,y,x) \in \Omega, \\ u|_{t<0} = 0 & , \end{cases}$$

où

$$\mathcal{H} := \partial_t + \sum_{j=1}^{d} A_j(t,y,x)\partial_j,$$

et avec $\Omega = \{(t,y,x) \in (0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}\}$, $T > 0$ étant fixé une fois pour toutes. Je note :

$$\mathcal{H}^{\pm} := \partial_t + \sum_{j=1}^{d} A_j^{\pm}(t,y,x)\partial_j,$$

où $A_j^{\pm}$ est la restriction de A_j à $\pm x > 0$.
L'inconnue du problème, $u(t,y,x)$ appartient à $\mathbb{R}^N$ et les matrices A_j appartiennent à $\mathcal{M}_N(\mathbb{R})$. Je suppose que les coefficients A_j sont constants en dehors d'un compact, C^∞ par morceaux, et que la discontinuité du coefficient est localisée sur l'hypersurface d'équation $x = 0$.
Les matrices $B_{j,k}$ dépendent de manière C^∞ de (t,y,x) et sont constantes en dehors d'un compact.
Le terme source f appartient à $H^\infty((0,T)\times\mathbb{R}^d)$, et est tel que $f|_{t<0} = 0$.

L'une des difficultés majeures, vis-à-vis de l'interprétation du problème, réside dans la définition du produit non-conservatif: $A_d\partial_x u$, dans le cas où u est discontinue en $\{x = 0\}$. En revanche, ce problème ne se pose plus lorsque je considère le problème visqueux, parabolique à $\varepsilon > 0$ fixé, suivant:

$$\begin{cases} \mathcal{H}^\varepsilon u^\varepsilon = f, & (t,y,x) \in \Omega, \\ u^\varepsilon|_{t<0} = 0, \end{cases} \tag{1.2.1}$$

où la pertubation visqueuse de l'opérateur hyperbolique $\mathcal{H}$ que j'envisage est donnée par:

$$\mathcal{H}^\varepsilon := \partial_t + \sum_{j=1}^{d-1} A_j\partial_j + A_d\partial_x - \varepsilon \sum_{1\leq j,k\leq d} \partial_j(B_{j,k}\partial_k.)$$

Les hypothèses faites ici, inspirées de celles faites lors de l'étude visqueuse des chocs, se découpent en des hypothèses de structure et des hypothèses géométriques. Commençons par les hypothèses structurelles. Tout d'abord, précisons l'hypothèse d'hyperbolicité pour $\mathcal{H}$, ainsi que l'hypothèse d'hyperbolicité-parabolicité qui assure la compatibilité entre la parabolicité de $\mathcal{H}^\varepsilon$ pour $\varepsilon > 0$ et l'hyperbolicité de $\mathcal{H} = \mathcal{H}^\varepsilon|_{\varepsilon=0}$.

Hypothèse 1.2.1 (Hyperbolicité à multiplicité constante de $\mathcal{H}$)**.**

Pour tout $(t, y, x) \in (0, T) \times \mathbb{R}^{d-1} \times \mathbb{R}^*$ *et* $(\eta, \xi) \neq 0_{\mathbb{R}^d}$*, la matrice*

$$\sum_{j=1}^{d-1} \eta_j A_j(t, y, x) + \xi A_d(t, y, x)$$

doit être diagonalisable sur $\mathbb{R}$*. De plus, ses valeurs propres ont une multiplicité constante.*

Le symbole de la partie parabolique, B, est défini par:

$$B(t, y, x, \eta, \xi) := \sum_{j,k<d} \eta_j \eta_k B_{j,k}(t, y, x)$$

$$+ \sum_{j<d} \xi \eta_j (B_{j,d}(t, y, x) + B_{d,j}(t, y, x)) + \xi^2 B_{d,d}(t, y, x).$$

Je suppose la condition de Majda et Pego ([MP85]) satisfaite ; on a donc :

Hypothèse 1.2.2 (Hyperbolicité-Parabolicité de $\mathcal{H}^\varepsilon$)**.**
Il existe $c > 0$ *tel que pour tout* $(t, y, x) \in (0, T) \times \mathbb{R}^{d-1} \times \mathbb{R}^*$ *et* $(\eta, \xi) \in \mathbb{R}^d$*, les valeurs propres de la matrice*

$$i \left(\sum_{j=1}^{d-1} \eta_j A_j(t, y, x) + \xi A_d(t, y, x) \right) + B(t, y, x, \eta, \xi)$$

vérifient $\Re e\ \mu \geq c(|\eta|^2 + \xi^2)$.

Je suppose ensuite que l'hypersurface sur laquelle se produit la discontinuité du coefficient, d'équation $x = 0$, est non-caractéristique pour l'opérateur $\mathcal{H}$, ce qui signifie :

Hypothèse 1.2.3 (Bord non-caractéristique)**.**
$\forall t \in (0, T)$, det $A_d|_{x=0^+}(t) \neq 0$ et det $A_d|_{x=0^-}(t) \neq 0$.

Les deux hypothèses géométriques (signe et transversalité) que je vais introduire maintenant apparaissent naturellement dans l'étude des chocs qui est un problème mathématiquement très voisin. Dans le cas que je traite, on verra que cette hypothèse empêche l'apparition de modes expansifs. La notion d'expansivité de la discontinuité pour des systèmes non diagonaux sera dûment explicitée lors du chapitre suivant. Je noterai $A_d^{\pm}$ la restriction de A_d à $\{\pm x > 0\}$.

Hypothèse 1.2.4 (Hypothèse de signe)**.**
Il existe $p \leq N - 1$ et $q \geq 0$ tels que :

- *Les valeurs propres de la matrice $A_d^-(t, y, 0)$, ordonnées par ordre croissant notées par $(\lambda_i^-(t, y))_{1 \leq i \leq N}$, sont telles que $\lambda_p^- < 0$ et $\lambda_{p+1}^- > 0$.*
- *Les valeurs propres de la matrice $A_d^+(t, y, 0)$, ordonnées par ordre croissant notées par $(\lambda_i^+(t, y))_{1 \leq i \leq N}$, vérifient $\lambda_{p+q}^+ < 0$ et $\lambda_{p+q+1}^+ > 0$.*

L'Hypothèse 1.2.4 interdit en particulier le cas où $A_d^+(t, y, 0)$ a toutes ses valeurs propres positives et $A_d^-(t, y, 0)$ a toutes ses valeurs propres négatives. Cette hypothèse ne suffit pas à interdire l'apparition de modes expansifs. Cette notion de modes expansifs est claire pour des matrices diagonales (on est ramené dans ce cas à la définition donnée lors du chapitre précédent pour des équations scalaires). On se place par exemple dans le cas des systèmes 2×2, avec $B_{d,d} = Id$ (de même que pour la régularisation visqueuse des équations scalaires du chapitre précédent) et on considère un coefficient $A_d := A_d^+ \mathbf{1}_{x>0} + A_d^- \mathbf{1}_{x<0}$ diagonal et constant de part et d'autre de $\{x = 0\}$. Si A_d^- et A_d^+ ont toutes deux une valeur propre < 0 et une valeur propre > 0, l'hypothèse de signe est bien vérifiée. Pourtant, deux cas de figure satisfont ces hypothèses. Prenons, par exemple,

$$A_d^- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Alors, si

$$A_d^+ = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},$$

deux modes traversants sont présents. Par contre, si

$$A_d^+ = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},$$

je me trouve dans le cas d'un mode compressif et d'un mode expansif. L'hypothèse de transversalité donnée ci-dessous interdit la présence de modes expansifs dans l'exemple qui vient d'être décrit.
Soit $G_d := (B_{d,d})^{-1} A_d$. L'espace vectoriel $\mathbb{E}_-(G_d)$ [resp $\mathbb{E}_+(G_d)$] est défini comme l'espace généré par les vecteurs propres associés aux valeurs propres < 0 [resp > 0] de G_d. L'hypothèse de transversalité s'écrit alors :

Hypothèse 1.2.5 (Transversalité)**.**
Les espaces $\mathbb{E}_-(G_d|_{x=0^+})$ et $\mathbb{E}_+(G_d|_{x=0^-})$ s'intersectent transversalement dans $\mathbb{R}^N$, ce qui s'exprime également ainsi:

$$\mathbb{E}_-(G_d|_{x=0^+}) + \mathbb{E}_+(G_d|_{x=0^-}) = \mathbb{R}^N.$$

Je vais maintenant donner l'hypothèse de stabilité, qui est de nature géométrique. Pour cela, je vais au préalable introduire quelques notations. Dans ce qui suit, $\eta := (\eta_1, \ldots, \eta_{d-1})$ sera la variable Fourier duale de y et ξ la variable Fourier duale de x. De plus, γ servira à noter un paramètre ≥ 0. Le paramètre ζ sera alors défini par $\zeta := (\tau, \gamma, \eta)$.
Notons $\mathbb{A}^\pm$ la matrice de $\mathcal{M}_{2N}(\mathbb{C})$ donnée par :

$$\mathbb{A}^\pm(t, y, x; \zeta) = \begin{pmatrix} 0 & Id \\ \mathcal{M}^\pm(t, y, x; \zeta) & \mathcal{A}^\pm(t, y, x; \eta) \end{pmatrix},$$

avec

$$\mathcal{M}^\pm(t, y, x; \zeta) = B_{d,d}^{-1} A_d^\pm(t, y, x) A^\pm(t, y, x; \zeta) + B_{d,d}^{-1}(t, y, x) \sum_{j,k=1}^{d-1} \eta_j \eta_k B_{j,k}(t, y, x),$$

où $A^\pm$ est le symbole hyperbolique tangentiel défini par :

$$A^\pm(t, y, x; \zeta) := (A_d^\pm)^{-1}(t, y) \left((i\tau + \gamma) Id + \sum_{j=1}^{d-1} i\eta_j A_j(t, y, x) \right).$$

Finalement, notons :

$$\mathcal{A}^{\pm}(t,y,x;\eta) = B_{d,d}^{-1}A_d^{\pm}(t,y,x) - B_{d,d}^{-1}(t,y,x)\sum_{j=1}^{d-1} i\eta_j\left(B_{j,d}(t,y,x) + B_{d,j}(t,y,x)\right).$$

J'introduis le poids $\Lambda(\zeta)$ utilisé pour une remise à l'échelle quand j'étudie le comportement haute fréquence, c'est-à-dire pour $|\zeta|$ grand :

$$\Lambda(\zeta) = \left(1 + \tau^2 + \gamma^2 + |\eta|^4\right)^{\frac{1}{4}}.$$

L'application J_Λ est définie de $\mathbb{C}^N \times \mathbb{C}^N$ dans $\mathbb{C}^N \times \mathbb{C}^N$ par :

$$(u,v) \mapsto (u, \Lambda^{-1}v).$$

Les espaces positifs et négatifs des matrices $\mathbb{A}^{\pm}(t,y,x;\eta)$, remis à l'échelle sont définis par :

$$\widetilde{\mathbb{E}}_{\pm}(\mathbb{A}^{\pm}) := J_\Lambda \mathbb{E}_{\pm}(\mathbb{A}^{\pm}).$$

L'hypothèse de stabilité, de nature spectrale, s'écrit alors :

Hypothèse 1.2.6 (Condition d'Evans uniforme)**.**
Supposons que pour tout $(t,y) \in (0,T) \times \mathbb{R}^{d-1}$ *et* $\zeta = (\tau,\eta,\gamma) \in \mathbb{R}^d \times \mathbb{R}^+ - \{0_{\mathbb{R}^{d+1}}\}$, *on a:*

$$\widetilde{D}(t,y,\zeta) = \left| det\left(\widetilde{\mathbb{E}}_-(\mathbb{A}^+(t,y,0;\zeta)), \widetilde{\mathbb{E}}_+(\mathbb{A}^-(t,y,0;\zeta))\right)\right| \geq C > 0.$$

Le déterminant de deux espaces vectoriels se calcule en choisissant une base orthonormée directe pour chacun d'entre eux. L'hypothèse de stabilité introduite ci-dessus ne dépend pas, bien sûr, du choix de ces bases. Les zéros de $\widetilde{D}$ représentent les fréquences pour lesquelles le problème symbolique associé est instable. Ce type d'hypothèse a été dégagé lors de travaux sur les ondes de choc. On peut notamment se référer aux travaux de D. Serre et K. Zumbrun ([SZ01],[ZS99]), de S. Benzoni, D. Serre et K. Zumbrun ([BGSZ06],[BGSZ01]), de F. Rousset [Rou03], ainsi qu'aux travaux de O. Guès, G. Métivier, K. Zumbrun et M. Williams ([GMWZ05] par exemple), de G. Métivier et K. Zumbrun ([MZ05], [MZ04]) et au livre de G. Métivier ([Mét04]).

Sous ces hypothèses, pour tout $\varepsilon > 0$ fixé, le problème parabolique (1.2.1) a une unique solution u^ε. Cette solution appartient à $H^\infty((0,T)\times$

$\mathbb{R}^{d-1} \times \mathbb{R}^*$); de plus, elle appartient globalement à $C^1((0,T) \times \mathbb{R})$. Si l'on note $u^{\varepsilon\pm}$ la restriction de u^ε à $\{\pm x > 0\}$, u^ε satisfait les conditions de transmission à l'interface :

$$\begin{cases} u^{\varepsilon+}|_{x=0^+} - u^{\varepsilon-}|_{x=0^-} = 0, \\ \partial_x u^{\varepsilon+}|_{x=0^+} - \partial_x u^{\varepsilon-}|_{x=0^-} = 0 \quad . \end{cases}$$

Je montre que les conditions au bord résiduelles obtenues sur u s'écrivent alors:

$$u|_{x=0^+} - u|_{x=0^-} \in \Sigma,$$

où Σ est un sous-espace de $\mathbb{R}^N$, dépendant du choix du tenseur de viscosité.
Le sous-espace Σ est donné par :

$$\Sigma := \left((G_d|_{x=0^+})^{-1} - (G_d|_{x=0^-})^{-1}\right) \left(\mathbb{E}_-(G_d|_{x=0^+}) \bigcap \mathbb{E}_+(G_d|_{x=0^-})\right).$$

Je prouve ainsi le Théorème 3.2.9 de la Thèse, que l'on rappelle ici:

Théorème 1.2.1. *Il existe $C > 0$ tel que, pour tout $0 < \varepsilon < 1$,*

$$\|u^\varepsilon - \underline{u}\|_{L^2(\Omega)} \leq C\varepsilon,$$

où u^ε est la solution de 1.2.1 et $\underline{u} := \underline{u}^+ \mathbf{1}_{x>0} + \underline{u}^- \mathbf{1}_{x<0}$ la solution du problème de transmission bien posé suivant:

$$\begin{cases} \mathcal{H}^+ \underline{u}^+ = f^+, & (t,y,x) \in (0,T) \times \mathbb{R}^d_+, \\ \mathcal{H}^- \underline{u}^- = f^-, & (t,y,x) \in (0,T) \times \mathbb{R}^d_-, \\ \underline{u}|_{x=0^+} - \underline{u}|_{x=0^-} \in \Sigma, \\ \underline{u}|_{t<0} = 0 \quad . \end{cases}$$

Cette preuve se fait sans avoir recours au calcul pseudo-différentiel dans le cas où les coefficients sont constants de part et d'autre de $\{x = 0\}$. Dans ce cas, je prouve d'abord des estimations, en variables de Fourier, par symétriseur de Kreiss, qui donnent ensuite l'estimation voulue via le théorème de Fourier-Plancherel.

Ce théorème montre que, pour un tenseur de viscosité donné, l'approche à viscosité évanescente sélectionne une solution. La dimension de l'espace vectoriel Σ (indépendante des variables tangentielles) exprime le nombre de modes compressifs présents dans la discontinuité.

Si la discontinuité n'a que des modes traversants, alors $\Sigma = \{0\}$, et donc la solution $\underline{u}$ sélectionnée appartient à $C^0((0,T) \times \mathbb{R}^d)$ mais n'appartient pas à $C^1((0,T) \times \mathbb{R}^d)$. Quand des modes compressifs sont présents, $\underline{u}$ est en général discontinue sur l'hypersurface d'équation $x = 0$.
Dans tous les cas, la solution $\underline{u}$ obtenue appartient à $H^\infty((0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}^*)$; on n'a donc aucune perte de régularité sur les demi-espaces $\{x > 0\}$ et $\{x < 0\}$ en passant à la limite visqueuse. Cela montre que les seules couches limites présentes se forment le long de l'hypersurface non-caractéristique $\{x = 0\}$.
En l'absence de modes compressifs, les couches limites formées sont de faible amplitude.

Remarquons que l'étude du problème dans sa formulation non-conservative fournit des résultats différents de ceux obtenus par l'étude des problèmes pris dans leur forme conservative. En effet, comme je l'ai montré dans le chapitre précédent, pour une formulation conservative du problème, la présence de modes compressifs induit la formation d'une masse de Dirac, quand $\varepsilon \to 0^+$. A contrario, les seules singularités observées ici sont des sauts de la fonction ou de ses dérivées.

J'ai choisi d'imposer que $u|_{t<0} = 0$ et que $f|_{t<0}$, afin que les conditions de compatibilité soient trivialement satisfaites. Si ce n'était pas le cas, pour chaque mode traversant, une singularité se formerait en $(t = 0, x = 0)$ puis se propagerait le long des caractéristiques issues de ce point. Supposer que les conditions de compatibilité sont satisfaites permet d'isoler les singularités provoquées par les discontinuités de coefficients.

1.2.2 Traitement du cas scalaire expansif.

Mon but a été d'obtenir un début de réponse concernant l'intégration de modes expansifs à l'approche précédente. Un premier pas est en effet d'étendre les techniques d'estimations d'énergie précédemment employées (preuve par transformée de Fourier, puis construction d'un symétriseur de type Kreiss) au cas expansif le plus simple : le cas scalaire 1-D à coefficients constants par morceaux. Je ne suppose aucune condition de compatibilité satisfaite lors de cette étude. Auparavant, les conditions de compatibilité servaient à éviter l'apparition d'un type de singularités non intrinsèquement lié à la discontinuité du coefficient. Le résultat obtenu ici montre que, dans le cas expansif, ce sont justement ces singularités-là qui sont induites par la discontinuité du coefficient. Ainsi, j'étudie indirectement la structure des couches limites se formant dans un cas traversant si les hypothèses de compatibilité des données ne sont pas satisfaites.

Soit $a(x) = a_R \mathbf{1}_{x>0} + a_L \mathbf{1}_{x<0}$, où a_R est une constante > 0 et a_L est une constante < 0 (expansivité de la discontinuité).Considérons alors le problème visqueux, parabolique à $\varepsilon > 0$ fixé, suivant :

$$\begin{cases} \partial_t u^\varepsilon + a(x)\partial_x u^\varepsilon - \varepsilon \partial_x^2 u^\varepsilon = f, & x \in \mathbb{R}, \\ u^\varepsilon|_{t=0} = h \quad , \end{cases} \tag{1.2.2}$$

où f et h désignent deux fonctions C^∞ et à support compact; mon résultat, énoncé ci-après, montre la convergence de u^ε vers une certaine fonction $\underline{u}$, quand $\varepsilon \to 0^+$. Il s'agit là d'un résultat analogue à celui établi lors du chapitre précédent, mais sa démonstration est tout à fait différente.

La solution du problème limite $\underline{u}$ est donnée par:

$$\underline{u} := u_R \mathbf{1}_{x\geq 0} + u_L \mathbf{1}_{x<0},$$

où u_L est la solution du problème mixte hyperbolique suivant:

$$\begin{cases} \partial_t u_L + a_L \partial_x u_L = f_L, & \{x < 0\}, \\ u_L|_{x=0} = h_L(0) + \displaystyle\int_0^t f|_{x=0}(s)\ ds, & \forall t \in (0,T). \\ u_L|_{t=0} = h_L \quad , \end{cases}$$

et u_R est la solution du problème mixte hyperbolique suivant:

$$\begin{cases} \partial_t u_R + a_R \partial_x u_R = f_R, & \{x > 0\}, \\ u_R|_{x=0} = h_R(0) + \displaystyle\int_0^t f|_{x=0}(s)\ ds, & \forall t \in (0,T), \\ u_R|_{t=0} = h_R \quad , \end{cases}$$

avec f_R [resp h_R] désignant la restriction de la fonction f [resp h] au demi-espace $\{x > 0\}$, et f_L [resp h_L] servant à noter la restriction de la fonction f [resp h] au demi-espace $\{x < 0\}$. Mon résultat, qui constitue le Théorème 3.3.2 de la Thèse, s'écrit:

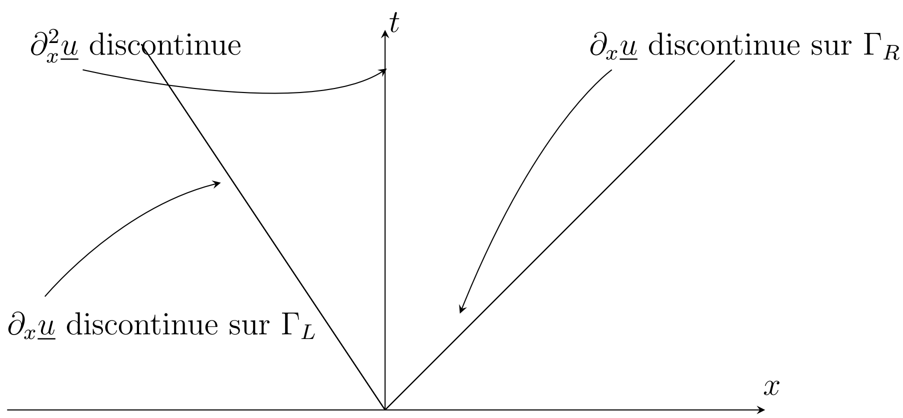
Théorème 1.2.2. *Il existe* $C > 0$ *tel que, pour tout* $0 < \varepsilon < 1$, *on ait:*

$$\|u^\varepsilon - \underline{u}\|_{L^2((0,T)\times\mathbb{R})} \leq C\varepsilon,$$

où u^ε *désigne la solution de* (1.2.2).

FIGURE 5

Singularités de la solution à petite viscosité, dans le cas scalaire expansif non-conservatif



La fonction $\underline{u}$ appartient ici à $C^0((0,T)\times\mathbb{R}) \bigcap L^2((0,T)\times\mathbb{R})$. Cependant $\underline{u} \notin C([0,T] : H^s(\mathbb{R}))$, $\forall s > \frac{3}{2}$. Ceci s'explique par les singularités de contact, localisées le long des courbes caractéristiques Γ_R et Γ_L issues du point $(t = 0, x = 0)$.

Une fois de plus, le résultat obtenu montre que l'approche visqueuse parvient à sélectionner une solution unique.

1.3 Une approche visqueuse pour des systèmes linéaires hyperboliques à coefficients discontinus incluant des modes expansifs (Chapitre 4).

Je généralise ici les résultats exposés lors du chapitre précédent à des discontinuités pouvant présenter des modes expansifs. Par souci de simplicité, je me suis restreint à des sytèmes à coefficients constants par morceaux en une dimension d'espace. Dans le même esprit que le chapitre précédent, je me ramène au problème de perturbation singulière suivant :

$$\begin{cases} \partial_t u^\varepsilon + A(x)\partial_x u^\varepsilon - \varepsilon \partial_x^2 u^\varepsilon = f, \quad (t,x) \in (0,T) \times \mathbb{R}, \\ u^\varepsilon|_{t=0} = h \quad , \end{cases}$$

où f et g sont C^∞ et à support compact. L'inconnue $u^\varepsilon(t,x)$ appartient à $\mathbb{R}^N$ et le coefficient A prend ses valeurs dans $\mathcal{M}_N(\mathbb{R})$.
Pour $\pm x > 0$, on a $A(x) = A^\pm$.
Comparativement aux hypothèses exposées au chapitre précédent, les hypothèses de signe et de transversalité sont remplacées ici par des hypothèses plus faibles qui autorisent les modes expansifs. De plus, dans le problème considéré maintenant, je ne fais aucune hypothèse de compatibilité entre f et h.

En pratique, j'ai préféré travailler sur la reformulation du problème visqueux en tant que problème de transmission.
La fonction $u^\varepsilon = u^{\varepsilon +}\mathbf{1}_{x>0} + u^{\varepsilon -}\mathbf{1}_{x<0}$ est en effet solution du problème suivant :

$$\begin{cases} \partial_t u^{\varepsilon +} + A^+ \partial_x u^{\varepsilon +} - \varepsilon \partial_x^2 u^{\varepsilon +} = f^+, \quad (t,x) \in (0,T) \times \mathbb{R}_+^*, \\ \partial_t u^{\varepsilon -} + A^- \partial_x u^{\varepsilon -} - \varepsilon \partial_x^2 u^{\varepsilon -} = f^-, \quad (t,x) \in (0,T) \times \mathbb{R}_-^*, \\ u^{\varepsilon +}|_{x=0^+} - u^{\varepsilon -}|_{x=0^-} = 0, \\ \partial_x u^{\varepsilon +}|_{x=0^+} - \partial_x u^{\varepsilon -}|_{x=0^-} = 0, \\ u^{\varepsilon \pm}|_{t=0} = h^\pm \quad . \end{cases} \tag{1.3.1}$$

Je vais maintenant exposer les différentes hypothèses que j'ai faites en commençant par écrire l'hypothèse d'hyperbolicité pour l'opérateur $\mathcal{H} := \partial_t + A\partial_x$:

Hypothèse 1.3.1 (Hyperbolicité)**.**
Les matrices A^+ et A^- sont constantes et diagonalisables dans $\mathbb{R}$.

Les hypothèses de parabolicité sont ici trivialement satisfaites. L'hypersurface $\{x = 0\}$ est supposée non-caractéristique pour $\mathcal{H}$, ce qui s'écrit:

Hypothèse 1.3.2 (Bord non-caractéristique)**.**
Les matrices A^+ et A^- sont inversibles.

L'hypothèse géométrique de stabilité s'écrit:

Hypothèse 1.3.3 (Condition d'Evans uniforme)**.**
Pour tout $\zeta = (\tau, \gamma) \in \mathbb{R} \times \mathbb{R}^+ - \{0_{\mathbb{R}^2}\}$, on a :

$$\left| det \left(\widetilde{\mathbb{E}}_-(\mathbb{A}^+(\zeta)), \widetilde{\mathbb{E}}_+(\mathbb{A}^-(\zeta)) \right) \right| \geq C > 0.$$

Je rappelle brièvement les notations employées ici. Les matrices $\mathbb{A}^\pm$ sont définies par :

$$\mathbb{A}^\pm = \begin{pmatrix} 0 & Id \\ (i\tau + \gamma) Id & A^\pm \end{pmatrix},$$

où $\mathbb{E}_+(\mathbb{A}^\pm)$ [resp $\mathbb{E}_-(\mathbb{A}^\pm)$] désigne l'espace vectoriel engendré par les vecteurs propres généralisés de $\mathbb{A}^\pm$ associés aux valeurs propres de $\mathbb{A}^\pm$ à partie réelle > 0 [resp < 0]. Le poids $\Lambda(\zeta)$ est donné par:

$$\Lambda(\zeta) = \left(1 + \tau^2 + \gamma^2\right)^{\frac{1}{2}}.$$

L'application J_Λ de $\mathbb{C}^N \times \mathbb{C}^N$ dans $\mathbb{C}^N \times \mathbb{C}^N$ se définit par :

$$(u, v) \mapsto (u, \Lambda^{-1} v).$$

On a alors :

$$\widetilde{\mathbb{E}}_\pm(\mathbb{A}^\pm) := J_\Lambda \mathbb{E}_\pm(\mathbb{A}^\pm).$$

Je vais maintenant donner quelques notations et propriétés nécessaires à la description de mon résultat principal.

Pour commencer, Σ est l'espace vectoriel défini par:

$$\Sigma := \left((A^+)^{-1} - (A^-)^{-1}\right) \left(\mathbb{E}_-(A^+) \bigcap \mathbb{E}_+(A^-)\right),$$

où j'ai noté, par exemple,

$$\mathbb{E}_-(A^+) = \bigoplus_{\lambda_j^+ < 0} \ker \left(A^+ - \lambda_j^+ Id\right),$$

avec les λ_j^+ représentant les valeurs propres de A^+, qui sont réelles et semi-simples de par l'hypothèse d'hyperbolicité.
J'introduis maintenant $\mathbb{I}$, qui est le sous-espace de $\mathbb{R}^N$ défini par :

$$\mathbb{I} := \mathbb{E}_-(A^-) \bigcap \mathbb{E}_+(A^+).$$

Le sous-espace $\mathbb{I}$, est, en ce qui concerne les modes expansifs, l'analogue de Σ pour les modes compressifs. En particulier, le nombre de modes expansifs est donné par $\dim \mathbb{I}$.
Choisissons, une fois pour toutes, un sous-espace vectoriel $\mathbb{V}$ de $\mathbb{R}^N$ satisfaisant :

$$\mathbb{E}_-(A^-) + \mathbb{E}_+(A^+) = \mathbb{I} \bigoplus \mathbb{V}.$$

Supposons que l'on a la décomposition suivante de $\mathbb{R}^N$:

$$\mathbb{R}^N = \mathbb{I} \bigoplus \mathbb{V} \bigoplus \Sigma.$$

Je note alors par $\Pi_{\mathbb{I}}$, $\Pi_{\mathbb{V}}$ et Π_{Σ} les projecteurs associés à cette décomposition de $\mathbb{R}^N$. L'application linéaire $\Pi_{\mathbb{I}}$ est donc le projecteur sur le sous-espace vectoriel $\mathbb{I}$ parallèlement au sous-espace vectoriel $\mathbb{V} \bigoplus \Sigma$.

Remarquons que, si l'on se replace dans le cadre des hypothèses géométriques données au chapitre précédent, alors $\mathbb{I} = \{0\}$. De plus, sous les hypothèses du chapitre précédent, on avait bien:

$$\mathbb{R}^N = \mathbb{V} \bigoplus \Sigma.$$

Sous mes nouvelles hypothèses, je prouve que, lorsque $\varepsilon \to 0^+$, la suite (u^ε) converge vers $\underline{u}$ dans $L^2((0,T)\times\mathbb{R})$, où $\underline{u} := u^+\mathbf{1}_{x\geq 0} + u^-\mathbf{1}_{x<0}$ est la solution du problème de transmission suivant :

$$\begin{cases} \partial_t u^- + A^-\partial_x u^- = f^-, & (t,x) \in (0,T)\times\mathbb{R}_-^*, \\ \partial_t u^+ + A^+\partial_x u^+ = f^+, & (t,x) \in (0,T)\times\mathbb{R}_+^*, \\ u^+|_{x=0} - u^-|_{x=0} \in \Sigma, \\ \partial_x \Pi_{\mathbb{I}} u^+|_{x=0} - \partial_x \Pi_{\mathbb{I}} u^-|_{x=0} = 0, \\ u^-|_{t=0} = h^-, \\ u^+|_{t=0} = h^+. \end{cases}$$

Dans ce problème, $f^\pm$ et $h^\pm$ désignent respectivement les restrictions des fonctions f et h au demi-espace $\{\pm x > 0\}$.

L'objet de la Proposition 4.2.12 de la Thèse est de montrer que ce problème est bien posé.

Une remarque s'impose: dès lors que $\mathbb{I} = \{0\}$, le problème hyperbolique limite quand $\varepsilon \to 0^+$ coincide avec celui identifié lors du chapitre précédent.

Pour en revenir aux hypothèses, je fais une hypothèse géométrique concernant la discontinuité de la matrice A. L'Hypothèse 1.3.4, que je vais introduire ici, généralise les hypothèses de transversalité et de signe du chapitre précédent.

Commençons par donner quelques notations préliminaires. De par l'hypothèse d'hyperbolicité, les matrices A^+ et A^- sont diagonalisables. Il existe donc deux matrices de passage P^+ et P^- et deux matrices diagonales D^+ et D^- telles que $D^+ = (P^+)^{-1}A^+P^+$ et $D^- = (P^-)^{-1}A^-P^-$.

Je définis alors le sous-espace $\mathbb{J}$ de $\mathbb{R}^N$ par

$$\mathbb{J} := \mathbb{E}_-(D^-) \bigcap \mathbb{E}_+(D^+).$$

Je choisis une fois pour toutes deux sous-espaces vectoriels de $\mathbb{R}^N$, $\mathbb{V}_1$ et $\mathbb{V}_2$, vérifiant :

$$\mathbb{V}_1 \bigoplus \mathbb{J} = \mathbb{E}_+(D^+),$$

et

$$\mathbb{V}_2 \bigoplus \mathbb{J} = \mathbb{E}_-(D^-).$$

Mon hypothèse, portant sur la discontinuité du coefficient, s'écrit alors:

Hypothèse 1.3.4 (Structure de la discontinuité)**.**
On a :

$$P^+\mathbb{V}_1 \bigoplus \left(P^+\mathbb{J} + P^-\mathbb{J}\right) \bigoplus P^-\mathbb{V}_2 \bigoplus \Sigma = \mathbb{R}^N$$

De plus, l'application

$$M := \begin{pmatrix} \Pi_{\mathbb{I}} P^+(D^+)^{-1} & -\Pi_{\mathbb{I}} P^-(D^-)^{-1} \\ P^+ & -P^- \end{pmatrix}$$

de $\mathbb{J}\times\mathbb{J}$ *dans* $\mathbb{I}\times(P^+\mathbb{J} + P^-\mathbb{J})$ *définit un isomorphisme entre les espaces* $\mathbb{J}\times\mathbb{J}$ *et* $\mathbb{I}\times\left(P^+\mathbb{J} + P^-\mathbb{J}\right)$.

Finalement, on a :

$$\dim\Big(\mathbb{E}_-(A^+)\bigcap\mathbb{E}_+(A^-)\Big) = \dim\Sigma.$$

Etant donné que cette hypothèse n'est pas vraiment aisée à vérifier, je présente ici une autre hypothèse plus simple: l'Hypothèse 1.3.5. L'Hypothèse 1.3.5 est une condition suffisante pour que l'Hypothèse 1.3.4 soit vérifiée.

Hypothèse 1.3.5 (Structure de la discontinuité, version suffisante)**.**
Supposons que :

- $\dim\Sigma = \dim\left(\mathbb{E}_-(A^+)\bigcap\mathbb{E}_+(A^-)\right)$
- $A^-\mathbb{I} = \mathbb{I}$
- $A^+\mathbb{I} = \mathbb{I}$
- $\ker((A^+)^{-1} - (A^-)^{-1})\bigcap\mathbb{I} = \{0\}$
- $\mathbb{E}_-((Id - \Pi_{\mathbb{I}})A^-(Id - \Pi_{\mathbb{I}}))\bigoplus\mathbb{E}_+((Id - \Pi_{\mathbb{I}})A^+(Id - \Pi_{\mathbb{I}}))\bigoplus\Sigma = \mathbb{V}\bigoplus\Sigma.$

Remarquons que, si A^- a toutes ses valeurs propres < 0 et A^+ a toutes ses valeurs propres > 0 (cas totalement expansif), cette hypothèse se réduit tout simplement à :

$$\ker\left((A^+)^{-1} - (A^-)^{-1}\right)\bigcap\mathbb{I} = \{0\},$$

ce qui est ici équivalent à :

$$det\left((A^+)^{-1} - (A^-)^{-1}\right) \neq 0.$$

En effet, dans le cas purement expansif, on a : $\mathbb{I} = \mathbb{R}^N$.
Sous ces hypothèses, je prouve le résultat suivant (Théorème 4.2.14) :

Théorème 1.3.1. *Il existe* $C > 0$ *tel que, pour tout* $0 < \varepsilon < 1$, *on ait:*

$$\|u^\varepsilon - \underline{u}\|_{L^2((0,T)\times\mathbb{R})} \leq C\varepsilon,$$

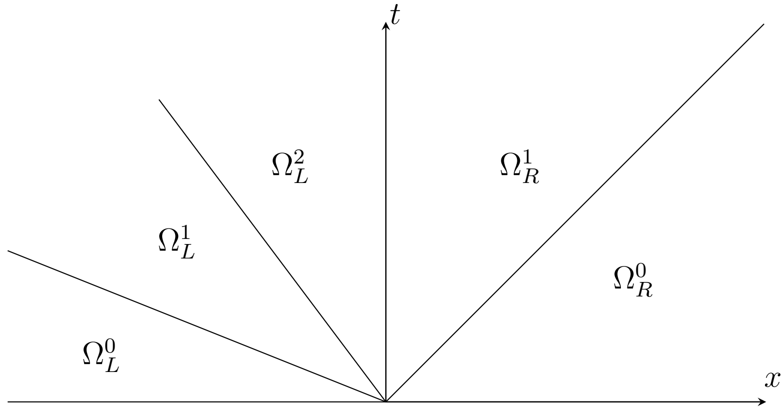
où u^ε désigne la solution du problème de transmission visqueux (1.3.1) *et $\underline{u}$ est définie comme étant l'unique solution du problème de transmission hyperbolique suivant :*

$$\begin{cases} \partial_t u^- + A^- \partial_x u^- = f^-, & (t,x) \in (0,T) \times \mathbb{R}_-^*, \\ \partial_t u^+ + A^+ \partial_x u^+ = f^+, & (t,x) \in (0,T) \times \mathbb{R}_+^*, \\ u^+|_{x=0} - u^-|_{x=0} \in \Sigma, \\ \partial_x \Pi_{\mathbb{I}} u^+|_{x=0} - \partial_x \Pi_{\mathbb{I}} u^-|_{x=0} = 0, \\ u^-|_{t=0} = h^-, \\ u^+|_{t=0} = h^+. \end{cases}$$

La preuve de ces estimations d'énergie est la même qu'au chapitre précédent. Pour obtenir ce résultat, ma principale préoccupation a été de trouver une solution approchée du problème de transmission visqueux (1.3.1). Dans ce cadre, je fournis un ansatz décrivant avec exactitude, à tout ordre, les couches limites se formant. Etant donné qu'aucune condition de compatibilité entre f et h n'est satisfaite, des couches limites caractéristiques se forment, en général, sur chacune des courbes caractéristiques issues de $(t,x) = (0,0)$.

Figure 6

Singularités de la solution à petite viscosité, dans le cas de sytèmes non-conservatifs



Ce dessin montre les zones où u est régulière dans le cas où $\dim \mathbb{E}_-(A^-) = 2$ et $\dim \mathbb{E}_+(A^+) = 1$.

Les couches limites se formant sur $\{x = 0\}$ sont de forte amplitude si $\mathbb{E}_-(A^+) \bigcap \mathbb{E}_+(A^-) \neq \{0\}$, et sont de faible amplitude dans le cas contraire.

En général, puisqu'aucune hypothèse de compatibilité des données n'est faite, il y a des singularités de la solution u (sauts de $\partial_x u$) localisées d'une part sur les courbes caractéristiques $\{(t, x) \in (0, T) \times \mathbb{R}_-^* : x - \lambda^- t = 0\}$, où λ^- désigne une valeur propre < 0 de A^- et d'autre part sur les courbes caractéristiques d'équation $\{(t, x) \in (0, T) \times \mathbb{R}_+^* : x - \lambda^+ t = 0\}$, où λ^+ représente une valeur propre > 0 de A^+.

Supposons, de plus, pour simplifier, que toutes les valeurs propres de A^+ et A^- sont distinctes (hypothèse de stricte hyperbolicité de l'opérateur $\partial_t + A\partial_x$), alors la singularité se produisant par exemple sur $\{(t, x) \in (0, T) \times \mathbb{R}_+^* : x - \lambda_1^+ t = 0\}$, où $\lambda_1^+ > 0$, est polarisée sur un espace vectoriel de dimension 1. Je me trouve donc ramené, après projection, à une analyse asymptotique semblable au cas scalaire expansif menée lors du chapitre précédent.

J'ai également explicité des exemples de systèmes satisfaisant l'ensemble des hypothèses décrites. Ceci s'avère important dans le cas présent, vu la quantité et la complexité des hypothèses mises en jeu. Il est à noter que les exemples exhibés ont été construits de manière à intégrer un mode expansif. J'ai montré, par exemple :

Proposition 1.3.6. *Soit P une matrice inversible de $\mathcal{M}_2(\mathbb{R})$, alors les matrices A^- et A^+ définies par :*

$$A^- = P^{-1} \begin{pmatrix} d_1^- & 0 \\ 0 & d_2^- \end{pmatrix} P$$

et

$$A^+ = P^{-1} \begin{pmatrix} d_1^+ & \alpha \\ 0 & d_2^+ \end{pmatrix} P$$

avec $d_1^- < 0$, $d_1^+ > 0$ et $\alpha \in \mathbb{R} - \{0\}$, satisfont l'ensemble des hypothèses faites si et seulement si, ou bien d_2^+ et d_2^- sont de même signe (strictement), ou bien $d_2^- < 0$ et $d_2^+ > 0$.

La preuve de cette proposition requiert une étude de la fonction d'Evans pour des sytèmes 2×2. La section 4.3 est d'ailleurs dédiée à cette étude.

Dans le cas où $d_2^- > 0$ et $d_2^+ < 0$, avec $\alpha \neq 0$, le problème reste ouvert. Il serait certainement intéressant de chercher à élucider ce cas.

1.4 Pénalisation de problèmes semi-linéaires avec condition au bord strictement maximale dissipative (Chapitre 5).

On s'intéresse ici à l'approximation de solutions de certains problèmes mixtes hyperboliques semi-linéaires, à bord caractéristique (à multiplicité constante) ou non-caractéristique. Les problèmes considérés sont symétriques avec une condition au bord strictement maximale dissipative. Ce travail fait l'objet d'un article co-écrit avec O. Guès. Les problèmes que nous étudions s'écrivent :

$$\begin{cases} Lu = F(t,x,u), \quad (t,x) \in (0,T) \times \Omega, \\ u|_{]]0,T[\times\partial\Omega} \in \mathcal{N}, \\ u|_{t=0} = 0 \quad , \end{cases}$$

où $L = A_0\partial_t + \sum_{j=1}^d A_j\partial_j + B$, les matrices A_j sont symétriques, de taille $N \times N$ et A_0 est uniformément définie positive. On suppose, de plus, que les matrices A_j dépendent de manière C^∞ de leurs paramètres et sont constantes en dehors d'un compact. La matrice B est aussi une matrice de $\mathcal{M}_N(\mathbb{R})$, dans C_b^∞ (il s'agit de l'ensemble des fonctions infiniment différentiables, bornées ainsi que toutes leurs dérivées), Ω est un ouvert de $\mathbb{R}^d$ à bord C^∞, $\mathcal{N}$ est un fibré vectoriel C^∞ sur $\mathbb{R} \times \partial\Omega$ définissant les conditions au bord, et F est une application C^∞ qui peut être non-linéaire.

Pour simplifier l'exposé, on va donner les résultats dans le cas où le problème est posé sur le demi-espace $\{x_d > 0\}$.
On notera alors par y la variable d'espace tangentielle donnée par $(x_1, \dots, x_{d-1})$; dans ce cas, le problème mixte hyperbolique s'écrit alors

$$\begin{cases} Lu = F(t,x,u), \quad (t,x) \in (0,T) \times \mathbb{R}_+^d, \\ u|_{x_d=0^+} \in \mathcal{N}(t), \quad \forall (t,y) \in (0,T) \times \mathbb{R}^{d-1}, \\ u|_{t=0} = 0. \end{cases}$$

Précisons les hypothèses concernant F : on prend $F \in C^\infty(\mathbb{R}^{1+d+N} : \mathbb{R}^N)$ telle que, pour tout $\alpha \in \mathbb{N}^{1+d+N}$, $\partial_{t,x,u}^\alpha F$ est bornée sur $\mathbb{R}^{1+d} \times K$ pour tout compact $K \subset \mathbb{R}^N$; de plus, $F(t,x,0) \in H^\infty(\mathbb{R}^{1+d})$ et $F|_{t<0} = 0$. Pour simplifier les choses, nous allons exposer nos résultats dans ce cadre.

On suppose que le bord $\{x_d = 0\}$ est à multiplicité constante :

Hypothèse 1.4.1. *La matrice de dérivée normale, A_d, a un rang constant, $N - d_0$, sur $\{x_d = 0\}$.*

Basiquement $d_0 := \dim \ker A_d|_{x_d=0}$. Si $d_0 = 0$, le bord est non-caractéristique. Cette hypothèse implique que $A_d|_{x_d=0}$ garde un nombre constant de valeurs propres > 0 et < 0. On notera $d_- := \dim \mathbb{E}_-(A_d|_{x_d=0})$ et $d_+ := \dim \mathbb{E}_+(A_d|_{x_d=0})$. On a la décomposition suivante de $\mathbb{R}^N$:

$$\mathbb{R}^N = \mathbb{E}_-(A_d) \bigoplus \mathbb{E}_+(A_d) \bigoplus \ker(A_d).$$

Dans ce qui suit, on notera respectivement $\mathbb{P}_+$, $\mathbb{P}_-$ et $\mathbb{P}_0$ les projecteurs associés à cette décomposition.

On suppose, de plus, que la condition au bord est strictement maximale dissipative, ce qui s'écrit :

Hypothèse 1.4.2 (Stricte maximale dissipativité du bord)**.**

$\mathcal{N}(t, y)$ est un sous-espace vectoriel de $\mathbb{R}^N$, de dimension $N - d^+$, dépendant de manière C^∞ de $(t, y) \in \mathbb{R}^d$, et il existe une constante $c_0 > 0$ telle que, pour tout $v \in \mathbb{R}^N$ et tout $(t, y) \in \mathbb{R}^d$, on ait :

$$v \in \mathcal{N}(t, y) \Rightarrow \langle A_d|_{x-d=0}(t, y)v, v\rangle \leq -c_0 \|(Id - \mathbb{P}_0)v\|^2.$$

Comme O. Guès l'a prouvé dans [Guè90], sous ces hypothèses, il existe $T_0 > 0$ tel que le problème mixte hyperbolique semi-linéaire considéré soit bien posé pour $T = T_0$.

Nous montrons que la solution de ce problème peut, par exemple, être approximée, quand $\varepsilon \to 0^+$, par la restriction à $x_d \geq 0$, de la solution d'un problème de Cauchy de la forme :

$$\begin{cases} L^\sharp u^\varepsilon + \frac{1}{\varepsilon} M u^\varepsilon \mathbf{1}_{x_d<0} = F^\sharp(t, x, u^\varepsilon), & (t, x) \in (0, T) \times \mathbb{R}^d, \\ \qquad\qquad\qquad\quad u^\varepsilon|_{t=0} = 0 & , \end{cases}$$

où $L^\sharp$ et $F^\sharp$ désignent des extensions de L et F à $\mathbb{R} \times \mathbb{R}^d$.

Ces extensions peuvent être choisies relativement arbitrairement. En fait, on peut prolonger les A_j avec $1 \leq j \leq d-1$, et B par des fonctions C^∞ jusqu'au bord pour $\{x_d \leq 0\}$, éventuellement discontinues en $\{x_d = 0\}$. Ces extensions seront notées respectivement $A_j^\sharp$ et $B^\sharp$. Concernant l'extension de L à $L^\sharp$, le point important est décrit par l'hypothèse suivante :

Hypothèse 1.4.3 (Continuité de $A_d^\sharp$)**.**

La matrice de dérivée normale, A_d, est la restriction à $\{x_d > 0\}$ d'une matrice $A_d^\sharp$ qui appartient à $C^\infty((0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}^*) \bigcap C^0(\mathbb{R}^{d+1})$ et qui est constante en dehors d'un compact.

Il est à noter que la discontinuité des coefficients ne pose pas de difficultés comme c'était le cas dans les chapitres 3 et 4, car, dans le cas présent, $A_d^\sharp$ est continue. Nous proposons deux approches :

- Dans la première approche, inspirée d'un travail de J. Rauch ([Rau79]) et d'un travail de C. Bardos et J. Rauch ([BR82]), la matrice M est définie positive (il s'agit de la matrice R donnée par (5.2.10)). Cela revient à pénaliser toutes les composantes. L'effet d'une telle pénalisation est "l'écrasement" de la solution obtenue sur $\{x_d < 0\}$. On obtient en effet,

  $$\lim_{\varepsilon \to 0^+} u^\varepsilon|_{t<0} = 0.$$

  D'un autre côté, nous avons choisi l'opérateur de pénalisation M de façon à obtenir la convergence de la suite u^ε vers une unique limite $\underline{u}$ satisfaisant :

  $$\lim_{x_d \to 0^+} \underline{u} := u|_{x_d=0^+}.$$

  Cela montre que même si, à $\varepsilon > 0$ fixé, u^ε est continue de part et d'autre de $\{x_d = 0\}$, $\underline{u}$ est en général discontinue en $\{x_d = 0\}$. Cela est révélateur de la présence de couches limites. Nous donnons ici notre résultat qui montre que les couches limites se formant sont localisées exclusivement sur le domaine fictif, ici à gauche de $\{x_d = 0\}$ (pour $x_d < 0$). Dans le cadre de notre première méthode, nous prouvons le résultat de convergence suivant (Théorème 5.2.6) :

Théorème 1.4.1. *Il existe $C > 0$ et $\varepsilon_0 > 0$ tel que, pour tout $0 < \varepsilon < \varepsilon_0$ on ait :*

$$\forall s > 0, \quad \|u^\varepsilon|_{x_d>0} - u\|_{H^s((0,T_0)\times\mathbb{R}^d_+)} \leq C\varepsilon$$

- Pour la deuxième approche, nous avons essayé de proposer une méthode qui ne génère pas de couches limites. D'un point de vue numérique, la présence de couches limites peut freiner la convergence vers la solution recherchée. Par exemple, dans l'article [PCLS05], A. Paccou, G. Chiavassa, J. Liandrat et K. Schneider observent numériquement un défaut de vitesse de convergence pour l'équation des Ondes. Dans l'annexe du chapitre 6 [section 6.6], je montre qu'il se forme effectivement des couches limites pour ce problème, ce qui répond à une question posée par les auteurs de [PCLS05], et explique la vitesse de convergence observée.

Le principe de la deuxième méthode de pénalisation de domaine que nous présentons ici est de "pénaliser exclusivement les modes sortants". Cela induit que l'opérateur de pénalisation M est ici une matrice positive au sens large, son noyau contenant les composantes de la solution qu'il n'est pas nécessaire de pénaliser (modes rentrants). L'expression précise de cette matrice est donnée par (5.2.15).

Par exemple, si $d^- = N$ (dans ce cas, tous les modes sont rentrants), alors $\mathcal{N} = \mathbb{R}^N$. La condition au bord $u|_{x_d=0} \in \mathcal{N}$ est alors systématiquement satisfaite et le problème considéré n'a pas besoin de condition au bord pour être bien posé. Ce problème n'a alors aucune nécessité d'être pénalisé. Cela signifie qu'il suffit de considérer que le bord est transparent (ce qui correspond à prendre $M = 0$) pour étendre notre problème à un problème de Cauchy posé sur tout l'espace.

A l'opposé, si $d^+ = N$, l'opérateur de pénalisation M que nous proposons est, dans ce cas, inversible.
Plus généralement, pour la méthode exposée ici, on a :

$$\dim\ker M = d_- + d_0.$$

Notre approche naît d'une remarque assez simple. Supposons que $\mathcal{N} = \mathbb{E}_-(A_d) \bigoplus \ker A_d$, alors, en prenant $M = \mathbb{P}_+$, on obtient le résultat suivant qui montre l'absence de formation de couches limites, à tout ordre :

Théorème 1.4.2. *Il existe $C > 0$ et $\varepsilon_0 > 0$ tel que, pour tout $0 < \varepsilon < \varepsilon_0$, d'une part on ait:*

$$\forall s > 0, \quad \|u^\varepsilon|_{x_d>0} - u\|_{H^s((0,T_0)\times\mathbb{R}^d_+)} \leq C\varepsilon.$$

D'autre part, il existe une unique fonction u^-, telle que

$$\forall s > 0, \quad \left\|u^\varepsilon|_{x_d<0} - u^-\right\|_{H^s((0,T_0)\times\mathbb{R}^d_-)} \leq C\varepsilon.$$

Cette fonction u^- vérifie $u^-|_{x_d=0} = u|_{x_d=0}$.

Ce résultat reste vrai dans le cas général (Théorème 5.2.7), même si $\mathcal{N} \neq \mathbb{E}_-(A_d) \bigoplus \ker A_d$, car, par un changement d'inconnue, on peut toujours se ramener au cas $\mathcal{N} = \mathbb{E}_-(A_d) \bigoplus \ker A_d$ (c'est le lemme 5.2.4); l'opérateur de pénalisation M obtenu est alors l'image inverse de $\mathbb{P}_+$ par ce changement d'inconnue. Ce résultat est meilleur que le précédent au point de vue de la qualité de convergence. En effet, la formation de couches limites, qui est une obstruction à la convergence, n'a pas du tout lieu ici, quel que soit l'ordre considéré (voir section 5.4.2).

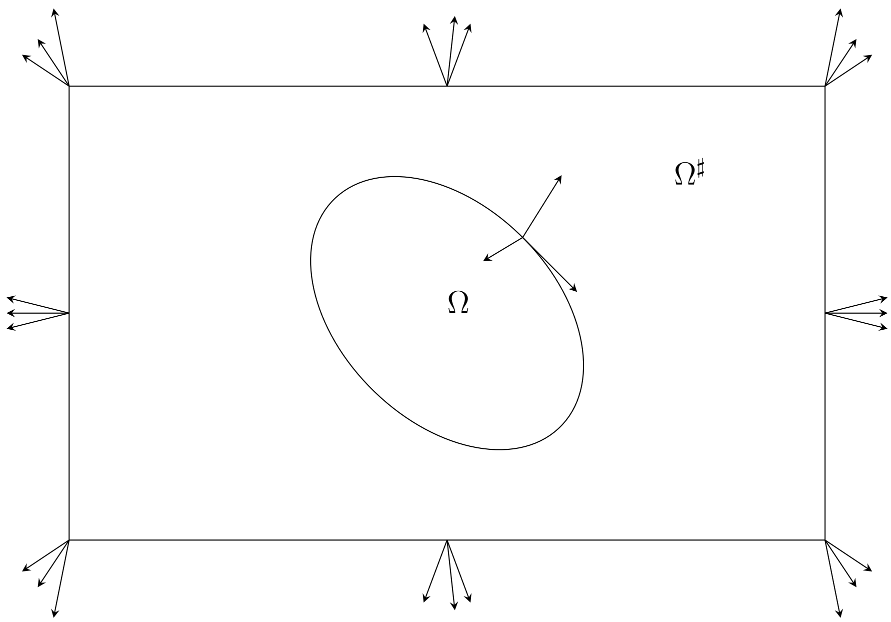
En pratique, il est plus commode de considérer un domaine fictif borné plutôt que le demi-espace $\{x_d < 0\}$. Soit l un réel > 0; par exemple, on peut approximer u par $u^\varepsilon|_{x_d>0}$, où u^ε est définie sur $(0,T)\times \mathbb{R}^{d-1} \times [-l,\infty)$ par :

$$\begin{cases} L^\sharp u^\varepsilon + \frac{1}{\varepsilon} M u^\varepsilon \mathbf{1}_{-L\leq x_d<0} = F^\sharp(t,x,u^\varepsilon), & (t,x) \in (0,T)\times\mathbb{R}^{d-1}\times[-L,\infty), \\ & u^\varepsilon|_{t=0} = 0. \end{cases}$$

On choisit ici $A_d^\sharp$ de sorte que $A_d^\sharp|_{x_d=-L}$ ait uniquement des valeurs propres < 0. Aucune condition au bord supplémentaire en $\{x_d = -L\}$ n'est donc nécessaire.

Comme cela est illustré ci-dessous, cette approche, consistant à introduire un bord absorbant, s'applique également à des domaines fictifs plus généraux, pouvant contenir des coins.

FIGURE 7
Domaine fictif à bord absorbant



Dans notre illustration, on a un mode sortant, un mode rentrant et un mode caractéristique sur $\partial\Omega$.
On prolonge l'opérateur de manière à ce que les trois modes soient sortants sur $\partial\Omega^\sharp$; ainsi aucune condition au bord n'est nécessaire sur $\partial\Omega^\sharp$.

1.5 Pénalisation de problèmes linéaires à coefficients constants satisfaisant une condition de Lopatinski uniforme (Chapitre 6).

L'objectif de ce chapitre est analogue au précédent, mais dans le cadre de conditions aux limites satisfaisant une condition de Lopatinski uniforme. Je considère un problème mixte hyperbolique linéaire du premier ordre, posé sur le demi-espace $\{x_d > 0\}$. En notant $y := (x_1, \ldots, x_{d-1})$

et $x := x_d$, ce problème s'écrit :

$$(1.5.1) \qquad \begin{cases} \mathcal{H}u = f, \quad \{x > 0\}, \\ \Gamma u|_{x=0} = \Gamma g, \\ u|_{t<0} = 0 \quad , \end{cases}$$

où l'inconnue $u(t, y, x)$ appartient à $\mathbb{R}^N$ et Γ est une application linéaire de rang p. On fixe $T > 0$, une fois pour toutes. On notera $\Omega^{\pm} := [0, T] \times \mathbb{R}^d_{\pm}$ et $\Upsilon := [0, T] \times \mathbb{R}^{d-1}$ où f désigne une fonction de $H^{\infty}(\Omega^+)$ et g est une fonction appartenant $H^{\infty}(\Upsilon)$. On suppose également que f et g sont telles que $f|_{t<0} = 0$ et $g|_{t<0} = 0$. L'opérateur $\mathcal{H}$ considéré est de la forme $\partial_t + \sum_{j=1}^{d} A_j \partial_j$ où les matrices A_j appartiennent à $\mathcal{M}_N(\mathbb{R})$ et sont constantes. Je fais l'hypothèse d'hyperbolicité suivante sur $\mathcal{H}$:

Hypothèse 1.5.1 (Hyperbolicité à multiplicité constante.)**.**

Pour tout $(\eta, \xi) \in \mathbb{R}^{d-1} \times \mathbb{R} - \{0\}$, *les valeurs propres de*

$$\sum_{j=1}^{d-1} \eta_j A_j + \xi A_d$$

sont réelles, semi-simples et de multiplicité constante.

Le bord, $\{x = 0\}$, est supposé non-caractéristique pour l'opérateur hyperbolique $\mathcal{H}$.

Hypothèse 1.5.2 (bord non-caractéristique)**.**

$$det A_d \neq 0.$$

Je suppose que l'opérateur de bord Γ satisfait avec $\mathcal{H}$ une Condition de Lopatinski Uniforme. Il s'agit là d'une condition de stabilité géométrique ([CP81]). H. O. Kreiss ([Kre70]) a prouvé que les problèmes mixtes strictement hyperboliques satisfaisant cette hypothèse sont bien posés. La condition d'Evans uniforme introduite précédemment est la version visqueuse de ce critère.

Hypothèse 1.5.3 (Condition de Lopatinski Uniforme)**.**

Pour tout ζ *tel que* $\gamma > 0$, *on a :*

$$|det(\mathbb{E}_-(A), \ker \Gamma)| \geq C > 0.$$

où A *est le symbole tangentiel de* $\mathcal{H}$, *défini par :*

$$A(\zeta) := -\left(A_d\right)^{-1}\left((i\tau + \gamma)Id + \sum_{j=1}^{d-1} i\eta_j A_j\right),$$

et $\zeta := (\gamma, \tau, \eta)$.

En particulier, j'ai montré, dans le chapitre 3, que la solution généralisée naturelle d'un problème hyperbolique linéaire discontinu de part et d'autre de $\{x = 0\}$ s'écrit $\underline{u} = \underline{u}^+ \mathbf{1}_{x>0} + \underline{u}^- \mathbf{1}_{x<0}$ de telle manière que la fonction U, définie sur $(0, T) \times \mathbb{R}^d_+$ par :

$$U(t, y, x) = \begin{pmatrix} \underline{u}^+(t, y, x) \\ \underline{u}^-(t, y, -x) \end{pmatrix}$$

soit la solution d'un problème mixte hyperbolique satisfaisant une condition de Lopatinski uniforme. Le point de vue est le même que celui du chapitre précédent.
Je vais montrer que l'on peut construire un multiplicateur de Fourier M et une fonction θ, tels que $u^\varepsilon|_{x>0}$ approxime la solution u de (1.5.1), quand $\varepsilon \to 0^+$, et ce, sans formation d'aucune couche limite; u^ε désigne ici la solution d'un problème de Cauchy, obtenu par perturbation singulière du problème (1.5.1), de la forme :

$$\begin{cases} \mathcal{H}^\sharp u^\varepsilon + \dfrac{1}{\varepsilon} M(\partial) \mathbf{1}_{x<0} u^\varepsilon = f^\sharp + \dfrac{1}{\varepsilon}\theta \mathbf{1}_{x<0}, & \{x \in \mathbb{R}\}, \\ u^\varepsilon|_{t<0} = 0 & . \end{cases} \tag{1.5.2}$$

Dans ce problème, $f^\sharp$ et $\mathcal{H}^\sharp$ sont des extensions de f et $\mathcal{H}$ à $\{x < 0\}$. En fait, étant donné que les coefficients de $\mathcal{H}$ sont constants, je prends $\mathcal{H}^\sharp := \mathcal{H}$. De plus, je prolonge f par 0 pour $x < 0$. J'ai mentionné que l'opérateur M de pénalisation est un multiplicateur de Fourier; cela signifie qu'il existe une matrice $M(\zeta)$ telle que

$$\mathcal{F}(M(\partial)u) = M(\zeta)\mathcal{F}(u),$$

où $\mathcal{F}$ désigne la transformée de Fourier tangentielle, c'est-à-dire par rapport aux variables (t, y). En s'inspirant des techniques utilisées lors du chapitre précédent (5), afin d'obtenir une pénalisation sans formation de couches limites, la matrice de $M(\zeta)$ possède en général un noyau bien choisi. Dans les approches présentées lors du chapitre précédent, nous avons pu tirer parti de la symétrie du problème assortie de l'hypothèse de dissipativité du bord afin de prouver mes estimations d'énergie.

Je présente ici deux solutions différentes au problème de pénalisation de domaine. Deux pénalisations adaptées (M_1, θ_1) (voir section 6.1.1) et (M_2, θ_2) (cf section 6.1.2) sont ainsi proposées. La deuxième manière de pénaliser semble plus avantageuse, dans une perspective de futures applications numériques, comme souligné ci-dessous. Ces deux manières de pénaliser, bien que très différentes, ont comme point commun de ne pas générer de couches limites, à tout ordre, ainsi qu'en témoigne le résultat suivant :

Théorème 1.5.1. *Considérons le problème* (1.5.2) *avec* $(M, \theta) = (M_1, \theta_1)$ *ou* $(M, \theta) = (M_2, \theta_2)$. *Alors, il existe* $C > 0$, *tel que, pour tout* $0 < \varepsilon < 1$ *et* $s \geq 0$, *on ait :*

$$\| u^\varepsilon|_{x>0} - u \|_{H^s(\Omega^+)} \leq C\varepsilon.$$

De plus, il existe une fonction $u^- \in H^\infty(\Omega^-)$ *et* $C' > 0$ *tels que pour tout* $0 < \varepsilon < 1$ *et* $s \geq 0$, *on ait :*

$$\| u^\varepsilon|_{x<0} - u^- \|_{H^s(\Omega^-)} \leq C'\varepsilon.$$

On a en outre :

$$u|_{x=0^+} = u^-|_{x=0^-}.$$

Première construction.

Une démarche tout à fait naturelle, dans le cas présent, est d'utiliser les symétriseurs intoduits par Kreiss, qui symétrisent le problème obtenu par transformée de Fourier tangentielle, tout en introduisant, pour ce problème, une propriété de dissipativité du bord. Il s'agit là de l'idée de base de ma première méthode de pénalisation de domaine.

Sous les hypothèses présentées ci-dessus, il est possible de construire un symétriseur de Kreiss. Le problème Kreiss-symétrisé, plus exactement sa transformée de Fourier-Laplace, est exactement analogue au problème traité lors du chapitre précédent. A la différence du problème traité lors du chapitre précédent, le problème obtenu ici a la fréquence ζ comme paramètre. Ma première approche se ramène en substance à construire un opérateur de pénalisation pour ce problème Fourier Kreiss-symétrisé. Comme je le montre section 6.2.1, modulo un changement de variable adéquat, l'opérateur de pénalisation est un projecteur sur l'espace négatif d'une matrice hermitienne, parallèlement à son espace positif.

L'orthogonalité de ce projecteur induit sa positivité, au sens large. Il s'agit là d'un point important dans la preuve de mes estimations d'énergie.

Le changement de variables évoqué passe par la construction d'une "matrice de Rauch", de même que les deux méthodes proposées lors du Chapitre 5 de la Thèse. Celui-ci est détaillé dans la section 6.2.2. La méthode de construction décrite ici se base sur la construction préliminaire d'un symétriseur de Kreiss S et d'une matrice de Rauch R, il est également à noter qu'il faut calculer les projecteurs orthogonaux sur l' espace $\mathbb{E}_-(R^{-1}SR^{-1})$. Ma première construction est détaillée dans la section 6.1.1.

Tirant parti du fait que l'opérateur considéré est à coefficients constants, mon résultat s'obtient finalement par le théorème de Fourier-Plancherel. Il est à remarquer que les estimations d'énergie obtenues ici sont prouvées en traitant le problème pénalisé (1.5.2) comme un problème de Cauchy sur tout l'espace (voir section 6.3.2).

Deuxième construction.

Ma deuxième construction est détaillée dans la section 6.1.2. Pour cette construction, je montre que les estimations, pour le problème de perturbation singulière considéré, résultent directement du fait que le problème de Cauchy (1.5.2) peut se reformuler comme un problème mixte hyperbolique satisfaisant une Condition de Lopatinski Uniforme. Sous mes hypothèses, cette condition de Lopatinski est trivialement vérifiée pour $\varepsilon > 0$ fixé. Je montre que c'est aussi le cas de la condition de Lopatinski obtenue asymptotiquement quand $\varepsilon \to 0^+$ (voir section 6.4.2). Une fois de plus, la clef de l'approche se situe dans une étude

du problème faite sur la transformée de Fourier-Laplace de l'équation. Le principe de mon approche est ici différent : au lieu de travailler à arranger l'opérateur, on reformule ici la condition au bord de manière plus adéquate (c'est le Lemme 6.4.1).
En se basant sur le fait que le problème considéré satisfait une condition de Lopatinski uniforme, je montre que la condition au bord $\Gamma u|_{x=0} = \Gamma g$ est équivalente, vis-à-vis de l'équation associée, à une autre condition au bord, directement adaptée à une approche par pénalisation de domaine. Pour le problème obtenu par transformée de Fourier-Laplace, l'opérateur de pénalisation alors prescrit, est le projecteur sur $\widetilde{\mathbb{E}}_-(A(\zeta))$ parallèlement à $\widetilde{\mathbb{E}}_+(A(\zeta))$, où il est à rappeler que A est le symbole tangentiel de $\mathcal{H}$ introduit précédemment. Ce projecteur sera noté $\mathbf{P}^-(\zeta)$. Les "tildes" sont utilisés pour indiquer qu'il s'agit des espaces étendus continûment à $\{\gamma = 0, (\tau, \eta) \neq 0\}$ ([CP81]). Contrairement à l'approche précédente, la positivité du projecteur n'est pas ici un facteur important pour la stabilité du problème. En revanche, il est primordial que, pour tout $\zeta \neq 0$, le noyau et l'image du projecteur $\mathbf{P}^-(\zeta)$ soient invariants par $A(\zeta)$.

D'un point de vue numérique, cette deuxième méthode de pénalisation présente l'avantage de nécessiter beaucoup moins de calculs que la première, qui fait intervenir le symétriseur de Kreiss, en général probablement (très?) difficile à calculer numériquement. Il est à noter que dans certains cas concrets (équation d'Euler par exemple), le symétriseur de Kreiss est en fait tout-à-fait aisé à calculer ; on pourra se référer au livre [BGS07] de S. Benzoni-Gavage et D. Serre.
De plus cette méthode est la seule méthode de pénalisation de la Thèse qui ne nécessite pas le calcul d'une matrice de Rauch.
En contrepartie, afin d'obtenir la fonction θ dans (1.5.2) pour ma deuxième méthode, il est nécessaire de calculer au préalable la solution v du problème de Cauchy :

$$\begin{cases} \mathcal{H}^\sharp v = f^\sharp, & \{x \in \mathbb{R}\}, \\ v|_{t<0} = 0 & , \end{cases}$$

ce qui n'est pas tellement contraignant, étant donné qu'aucun bord n'est présent.

Partie I:

Approches Visqueuses pour des Problèmes Hyperboliques à Coefficients Discontinus.

Chapter 2

Problèmes hyperboliques scalaires conservatifs à coefficients discontinus.

Ce chapitre contient le papier [For07c] intitulé "Two Results concerning the Small Viscosity Solution of Linear Scalar Conservation Laws with Discontinuous Coefficients" soumis à publication en juillet 2007.

Abstract

In this paper, we consider the vanishing viscosity approach of the linear hyperbolic Cauchy problem in 1-D

$$\begin{cases} \partial_t u + \partial_x (au) = f, & \{t > 0, x \in \mathbb{R}\}, \\ u_{|t=0} = h, \end{cases}$$

when the coefficient $a(t, x)$ is discontinuous across the line $\{x = 0\}$ and smooth on $\{x \neq 0\}$. Two cases are treated: the expansive (or completely outgoing) case where $sign\,(xa(t, x)) > 0$, for all (t, x) in a neighborhood of $\{x = 0\}$, and the compressive case (or completely ingoing) case where $sign\,(xa(t, x)) < 0$, for all (t, x) in a neighborhood of $\{x = 0\}$. In both cases, we show that the solution of the viscous problem converges and selects a well defined 'generalized solution'. In the expansive case, our first result answers the open question of selecting a unique solution to the hyperbolic problem, answering a question raised in paper [PR97]. In the compressive case, we show the formation of a Dirac measure in the small viscosity limit. Moreover, the considered problem does not need to be the linearized of a shock-wave on a shock front. For both results, a detailed asymptotic analysis is made via the construction of approximate solutions at any order, including a boundary

layer analysis. Moreover, both results state not only existence and uniqueness of the solution but its stability, and are new.

2.1 Introduction.

Consider the conservative 1-D Cauchy problem:

$$\begin{cases} \partial_t u + \partial_x (a(t,x)u) = f, & x \in \mathbb{R}, \\ u|_{t=0} = h & . \end{cases} \tag{2.1.1}$$

If a is discontinuous through $\{x = 0\}$, problem (2.1.1) has no classical sense and a new notion of solution has to be introduced. Several approaches have already been proposed. Among them, renormalized solutions for this sort of problems have been introduced by Diperna and Lions in [DL89]. A neighboring question is treated by LeFloch ([LeF90]), then generalized to 1-D systems by Hu and LeFloch ([HL96]). In [BJ98] and [BJM05], Bouchut, James and Mancini define a notion of solution around the parallel study of the conservative problem (2.1.1) and the associated nonconservative problem:

$$\begin{cases} \partial_t u + a(t,x)\partial_x u = g, & x \in \mathbb{R}, \\ u|_{t=0} = l & . \end{cases} \tag{2.1.2}$$

In [PR97], Poupaud and Rascle propose a notion of solution based on generalized characteristics in the sense of Filippov.

In this short paper, we will consider the vanishing viscosity approach in the case where $a(t,x)$ is a piecewise smooth function. Let us describe our assumptions. Let $T > 0$ be fixed once for all. We will assume that the coefficient a belongs to the space of infinitely differentiable functions, bounded with all their derivatives: $C_b^\infty([0,T] \times \mathbb{R}^*)$, with $\mathbb{R}^* = \mathbb{R} - \{0\}$. Furthermore, we assume that f belongs to $C_0^\infty([0,T] \times \mathbb{R})$ and h belongs to $C_0^\infty(\mathbb{R})$. As a first step, let us take $a(x) := a_R \mathbf{1}_{x>0} + a_L \mathbf{1}_{x<0}$, where a_L and a_R denote two constants in $\mathbb{R}^*$. Different cases have to be considered depending on the sign of a_L and a_R. Among those cases, the most interesting ones are when a_L and a_R are of opposite sign. If $a_L > 0$ and $a_R < 0$ [resp $a_L < 0$ and $a_R > 0$], the associated problem will fall into what we call the 'ingoing case' [resp 'outgoing case' or 'expansive case']. Our two results state existence, uniqueness and stability of the solution obtained by vanishing viscous perturbation of (2.1.1). The first result deals with the expansive case where uniqueness is the main concern whereas the second result deals with the ingoing case where existence is the main concern. Let ε denote a positive real number. Having in mind to make ε tends towards zero, we consider the following viscous perturbation of (2.1.1):

$$\begin{cases} \partial_t u^\varepsilon + \partial_x \left(a(t,x)u^\varepsilon\right) - \varepsilon \partial_x^2 u^\varepsilon = f, & x \in \mathbb{R}, \\ u^\varepsilon|_{t=0} = h & . \end{cases} \tag{2.1.3}$$

We prove then a convergence result stating that the solution u^ε of (2.1.3) tends towards $\underline{u}$ deduced from an asymptotic analysis of the problem. Naturally, $\underline{u}$ is then what could be called the small viscosity solution of (2.1.1). In the ingoing case, $\underline{u}$ is a measure-valued solution which coincides with the generalized solution introduced in the already cited papers. But the interesting point is the asymptotic expansion which gives a very precise description of the solution. In the expansive case, the result seems to be completely new, since the main difficulty was to 'select' a solution among all possible weak solutions.

2.2 Viscous treatment of the expansive case.

For our first result, let us consider equation (2.1.3) in the case where the coefficient a satisfies, for all $t \in [0,T]$,

$$a(t,0^+) > 0,$$

$$a(t,0^-) < 0.$$

We will denote by a_R the restriction of a to $\{x > 0\}$ and by a_L the restriction of a to $\{x < 0\}$.

Remark 2.2.1. *The value of $a|_{x=0}$ is of no concern here. Moreover, by taking $f = 0$, $a_L = -1$ and $a_R = 1$, we recover the singular expansive case given by Poupaud and Rascle as an example in [PR97].*

Let us define $\underline{u}$ by $\underline{u} := u_R \mathbf{1}_{x\geq 0} + u_L \mathbf{1}_{x<0}$, where (u_R, u_L) is the unique solution of the following problem:

$$\begin{cases} \partial_t u_R + \partial_x(a_R u_R) = f_R, & \{x > 0\}, \\ \partial_t u_L + \partial_x(a_L u_L) = f_L, & \{x < 0\}, \\ u_R|_{x=0} = u_L|_{x=0} = 0, \quad \forall t \in (0,T], \\ u_R|_{t=0} = h_R, u_L|_{t=0} = h_L & , \end{cases}$$

where f_R [resp h_R] denotes the restriction of f [resp h] to $\{x > 0\}$, and f_L [resp h_L] denotes the restriction of f [resp h] to $\{x < 0\}$. Note well that this problem has a unique solution in $L^2([0,T] \times \mathbb{R})$, which is given on the side $\{x < 0\}$ by:

$$\begin{cases} \partial_t u_L + \partial_x(a_L u_L) = f_L, & \{x < 0\}, \\ u_L|_{x=0} = 0, \quad \forall t \in (0,T], \\ u_L|_{t=0} = h_L \quad , \end{cases}$$

and on the side $\{x > 0\}$ by:

$$\begin{cases} \partial_t u_R + \partial_x(a_R u_R) = f_R, & \{x > 0\}, \\ u_R|_{x=0} = 0, \quad \forall t \in (0,T], \\ u_R|_{t=0} = h_R \quad . \end{cases}$$

Remark that, in general, $h_R(0) = h_L(0) \neq 0$, and thus the corner compatibilities are not satisfied. Let us compute $\underline{u}$ in the case where $f = 0$. We will first introduce some notations. Let Ω_R be $(0,T) \times \mathbb{R}^{*+}$. Consider now the vector field defined through: $(t,x) \mapsto \partial_t + a_R(t,x)\partial_x$. We will denote by Γ_R the characteristic curve passing through $t = 0, x = 0$ and tangent to this vector field. A parametrization of Γ_R is given by: $\Gamma_R = \{(t, x_R(t)), t \in (0,T)\}$, where x_R is the solution of the equation:

$$\begin{cases} \dfrac{dx_R}{dt}(t) = a_R(t, x_R(t)), & t \in (0,T), \\ x_R(0) = 0 \quad . \end{cases}$$

Let us denote by $\widetilde{a}_R$ an arbitrary smooth extension of a_R to $\{x < 0\}$. We define then φ_R as the solution of:

$$\begin{cases} (\partial_t + \widetilde{a}_R(t,x)\partial_x)\varphi_R = 0, & (t,x) \in (0,T) \times \mathbb{R}, \\ \varphi_R|_{t=0} = x \quad . \end{cases}$$

The obtained φ_R is in $C^\infty((0,T) \times \mathbb{R})$. Moreover, we have:

$$\Gamma_R = \{(t,x) \in \Omega_R : \varphi_R(t,x) = 0\}.$$

Ω_L, Γ_L and φ_L are defined in a symmetric way and there holds:

$$\Gamma_L = \{(t,x) \in \Omega_L : \varphi_L(t,x) = 0\}.$$

Note well that, by construction of φ_L and φ_R, we have:

Lemma 2.2.2. *There is c such that, for all $(t,x)\in\Gamma_R$, there holds:* $|\partial_x\varphi_R(t,x)|\geq c>0, \quad |\partial_x\varphi_L(t,x)|\geq c>0.$

Proof.

Differentiating the equation with respect to x, we obtain that $v := \partial_x\varphi_R$ is the solution of the following transport equation:

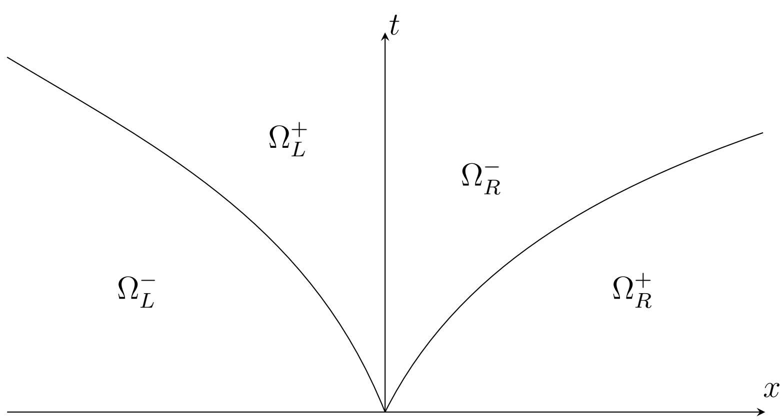
$$\begin{cases} (\partial_t+\widetilde{a}_R\partial_x)v+(\partial_x\widetilde{a}_R)v=0, & (t,x)\in(0,T)\times\mathbb{R}, \\ v|_{t=0}=1 & . \end{cases}$$

v is solution of a linear homogeneous equation thus, it cannot cancel without being identically equal to zero along the characteristic curve and in particular for $t=0$, which achieves to prove our Lemma for φ_R. The proof for φ_L is identical. □

We note for instance:

$$\Omega_L^+=\{(t,x)\in\Omega_L : \varphi_L(t,x)>0\},$$

where the 'L' stands for 'on left hand side of Γ_L' and the $+$ is related to the sign of $\varphi_L(t,x)$. We define in the same manner: Ω_L^-, Ω_R^+ and Ω_R^-.



Let us consider, as an example, the case where the coefficient is piecewise constant and $f=0$. Solving the limiting hyperbolic problem, we get that, for all $(t,x)\in\Omega_L^+\bigcup\Omega_R^-\bigcup\{x=0\}$,

$$\underline{u}(t,x)=0,$$

for all $(t,x)\in\Omega_R^+$,

$$\underline{u}(t,x)=h_R(x-a_Rt),$$

and for all $(t, x) \in \Omega_L^-$,

$$\underline{u}(t, x) = h_L(x - a_L t).$$

Observe that, in this case, the mass of $\underline{u}$ remains constant for all $t \in [0, T]$. Moreover, this example shows clearly the discontinuity of $\underline{u}$ through the lines $\{x - a_R t = 0\}$ and $\{x - a_L t = 0\}$.

Although equation (2.1.1) trivially admits an infinite number of solutions, we prove the following result:

Theorem 2.2.3. *There is $C > 0$ such that, for all $0 < \varepsilon < 1$, there holds:*

$$\|u^\varepsilon - \underline{u}\|_{L^\infty([0,T]:L^2(\mathbb{R}))} \leq C\varepsilon^{\frac{1}{4}},$$

where u^ε is the solution of (2.1.3).

Proof.
We will begin by constructing an approximate solution of problem (2.1.3). As a first step, we will reformulate problem (2.1.3) in an equivalent manner. The restrictions of u^ε to $\{x > 0\}$ and $\{x < 0\}$, denoted respectively by u_L^ε and u_R^ε satisfy the following transmission problem:

$$(2.2.1) \qquad \begin{cases} \partial_t u_R^\varepsilon + \partial_x(a_R u_R^\varepsilon) - \varepsilon\partial_x^2 u_R^\varepsilon = f_R, & \{x > 0\}, t \in [0, T], \\ \partial_t u_L^\varepsilon + \partial_x(a_L u_L^\varepsilon) - \varepsilon\partial_x^2 u_L^\varepsilon = f_L, & \{x < 0\}, t \in [0, T], \\ [u^\varepsilon]_{x=0} = 0, \\ [a(x)u^\varepsilon - \varepsilon\partial_x u^\varepsilon]_{x=0} = 0, \\ u_R^\varepsilon|_{t=0} = h_R, \\ u_L^\varepsilon|_{t=0} = h_L \quad . \end{cases}$$

Let us introduce $L_R^\varepsilon = \partial_t + \partial_x(a_R.) - \varepsilon^2\partial_x^2$ and $L_L^\varepsilon = \partial_t + \partial_x(a_L.) - \varepsilon^2\partial_x^2$. We perform the construction of the approximate solution separately on the four domains Ω_L^-, Ω_L^+, Ω_R^+ and Ω_R^-. . We will denote by $u_{app,L,+}^\varepsilon$ the restriction of u_{app}^ε to Ω_L^+ and so on. Let us present the different profiles and their ansatz:

$$u_{app,L,+}^\varepsilon(t, x) = \sum_{n=0}^{M} \left(\underline{\mathbf{U}}_{L,n,+}(t, x) + \mathbf{U}_{L,n,+}^c\left(t, \frac{\varphi_L(t, x)}{\sqrt{\varepsilon}}\right)\right)\varepsilon^{\frac{n}{2}},$$

where the profiles $\underline{\mathbf{U}}_{n,L,+}$ belongs to $H^\infty(\Omega_L^+)$ and the characteristic boundary layer profiles $\mathbf{U}_{n,L,+}^c(t, x, \theta_L)$ belongs to $e^{-\delta|\theta_L|}H^\infty((0, T) \times$

$\mathbb{R}^{*+})$, for some $\delta > 0$. We will take a similar ansatz for $u^{\varepsilon}_{app,L,-}$, $u^{\varepsilon}_{app,R,-}$ and $u^{\varepsilon}_{app,R,+}$ over their respective domains. Let us explain the different steps of the construction of the approximate solution. We begin by constructing the underlined profiles $\underline{\mathbf{U}}_n$ in cascade, the boundary layer profiles $\mathbf{U}^c_n$ are then computed as a last step. We construct our profiles such that, for all fixed $\varepsilon > 0$, u^{ε}_{app} belongs to $C^1([0,T]\times\mathbb{R})$. In what follows, we will note:

$$\underline{\mathbf{U}}_{R,j}(t,x) := \underline{\mathbf{U}}_{R,j,+}(t,x)\mathbf{1}_{(t,x)\in\Omega_R^+} + \underline{\mathbf{U}}_{R,j,-}(t,x)\mathbf{1}_{(t,x)\in\Omega_R^-}.$$

Moreover, we will note:

$$\mathbf{U}^c_{R,j}\left(t,x,\frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right) := \mathbf{U}^c_{R,j,+}\left(t,\frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right)\mathbf{1}_{(t,x)\in\Omega_R^+} + \mathbf{U}^c_{R,j,-}\left(t,\frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right)\mathbf{1}_{(t,x)\in\Omega_R^-}.$$

Note well that the dependence of $\mathbf{U}^c_{R,j}$ in x is a bit subtle. Actually, $\mathbf{U}^c_{R,j}$ is piecewise constant with respect to x on each side of Γ_R, which explains that $\mathbf{U}^c_{n,L,+}$ and $\mathbf{U}^c_{n,L,-}$ have no direct dependency in x. Due to their particular meaning, we prefer denoting the profiles $\underline{\mathbf{U}}_{R,0}$ and $\underline{\mathbf{U}}_{L,0}$ by u_R and u_L. Let us note $\mathcal{H}_R$ the differential operator

$$\mathcal{H}_R := \partial_t + \partial_x(a_R.)$$

and $\mathcal{P}_R$ the differential operator

$$\mathcal{P}_R := \partial_t + a_R\partial_x - (\partial_x\varphi)^2\partial^2_{\theta_R} + \partial_x a_R.$$

We have

$$L^{\varepsilon}_R\ u^{\varepsilon}_{R,app}\left(t,x,\frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right) = \sum_{j=0}^{M+1} L_{R,j}\left(t,x,\frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right)\varepsilon^{\frac{j}{2}}$$

where

$$L_{R,0} = \mathcal{H}_R u_R + \mathcal{P}_R U^c_{R,0},$$

$$L_{R,1} = \mathcal{H}_R\underline{\mathbf{U}}_{R,1} + \mathcal{P}_R U^c_{R,1} - \left(2(\partial_x\varphi)\partial_x\partial_{\theta_R} + (\partial^2_x\varphi)\partial_{\theta_R}\right)U^c_{R,0},$$

and, for $2 \le j \le M-1$, we get:

$$L_{R,j} = \mathcal{H}_R\underline{\mathbf{U}}_{R,j} + \mathcal{P}_R U^c_{R,j} - \left(2(\partial_x\varphi)\partial_x\partial_{\theta_R} + (\partial^2_x\varphi)\partial_{\theta_R}\right)U^c_{R,j-1} - \partial^2_x\underline{\mathbf{U}}_{R,j-2} - \partial^2_x U^c_{R,j-2},$$

$$L_{R,M} = \mathcal{P}_R U^c_{R,M} - \left(2(\partial_x\varphi)\partial_x\partial_{\theta_R} + (\partial^2_x\varphi)\partial_{\theta_R}\right)U^c_{R,M-1} - \partial^2_x\underline{\mathbf{U}}_{R,M-2} - \partial^2_x U^c_{R,M-2},$$

$$L_{R,M+1} = -\left(2(\partial_x\varphi)\partial_x\partial_{\theta_R} + (\partial_x^2\varphi)\partial_{\theta_R}\right) U^c_{R,M} - \partial_x^2\underline{\mathbf{U}}_{R,M-1} - \partial_x^2 U^c_{R,M-1}.$$

Symmetrically, there holds:

$$L_L^\varepsilon\ u^\varepsilon_{L,app}\left(t,x,\frac{\varphi_L(t,x)}{\sqrt{\varepsilon}}\right) = \sum_{j=0}^{M+1} L_{L,j}\left(t,x,\frac{\varphi_L(t,x)}{\sqrt{\varepsilon}}\right)\varepsilon^{\frac{j}{2}}$$

where, for instance, $L_{L,2}$ is given by:

$$L_{L,2} = \mathcal{H}_L\underline{\mathbf{U}}_{L,2} + \mathcal{P}_L U^c_{L,2} - \left(2(\partial_x\varphi_L)\partial_x\partial_{\theta_L} + (\partial_x^2\varphi_L)\partial_{\theta_L}\right) U^c_{L,1} - \partial_x^2 u_L - \partial_x^2 U^c_{L,0},$$

where $\mathcal{H}_L$ is defined by:

$$\mathcal{H}_L := \partial_t + \partial_x(a_L.)$$

and $\mathcal{P}_L$ is given by:

$$\mathcal{P}_L := \partial_t + a_L\partial_x - (\partial_x\varphi_L)^2\partial^2_{\theta_L} + \partial_x a_L.$$

Plugging $u^\varepsilon_{L,app}$ and $u^\varepsilon_{R,app}$ in the problem (2.2.1) and identifying the terms with the same scale in ε, making then $|\theta_L|$ and $|\theta_R|$ tend to infinity, we obtain the profiles equations satisfied by the underlined profiles. Let us begin by writing the equations satisfied by $\underline{\mathbf{U}}_{L,j}$ and $\underline{\mathbf{U}}_{R,j}$ for all $0 \le j \le M-1$. Thanks to the transmission conditions we had on the viscous problem, we get:

$$\begin{cases} u_{L,+}|_{x=0} - u_{R,-}|_{x=0} = 0, \\ a_L u_{L,+}|_{x=0} - a_R u_{R,-}|_{x=0} = 0. \end{cases}$$

This linear system being invertible, we get then the homogeneous Dirichlet boundary condition:

$$u_L|_{x=0} = u_R|_{x=0} = 0.$$

We can split these equations into three well-posed problems:

$$\begin{cases} \partial_t u_{R,-} + \partial_x(a_R u_{R,-}) = f_{R,-}, & (t,x) \in \Omega_R^-, \\ \partial_t u_{L,+} + \partial_x(a_L u_{L,+}) = f_{L,+}, & (t,x) \in \Omega_L^+, \\ u_{L,+}|_{x=0} = u_{R,-}|_{x=0} = 0, \end{cases}$$

$$\begin{cases} \partial_t u_{R,+} + \partial_x(a_R u_{R,+}) = f_{R,+}, & (t,x) \in \Omega_R^+, \\ u_R|_{t=0} = h_R \quad , \end{cases}$$

$$\begin{cases} \partial_t u_{L,-} + \partial_x(a_L u_{L,-}) = f_{L,-}, & (t,x) \in \Omega_L^-, \\ u_L|_{t=0} = h_L & . \end{cases}$$

Since these equations are well-posed, the function $\underline{u}$ is now perfectly defined. Let us go on with the construction of the next profiles. $U_{R,1}$ and $U_{L,1}$ are defined by:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{1,R,-} + \partial_x(a_R \underline{\mathbf{U}}_{1,R,-}) = 0, & (t,x) \in \Omega_R^-, \\ \partial_t \underline{\mathbf{U}}_{1,L,+} + \partial_x(a_L \underline{\mathbf{U}}_{1,L,+}) = 0, & (t,x) \in \Omega_L^+, \\ \underline{\mathbf{U}}_{1,L,+}|_{x=0} = \underline{\mathbf{U}}_{1,R,-}|_{x=0} = 0 & . \end{cases}$$

Thus $\underline{\mathbf{U}}_{1,R,-} = 0$ and $\underline{\mathbf{U}}_{1,L,+} = 0$.

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{1,R,+} + \partial_x(a_R \underline{\mathbf{U}}_{1,R,+}) = 0, & (t,x) \in \Omega_R^+, \\ \underline{\mathbf{U}}_{1,R,+}|_{t=0} = 0 & , \end{cases}$$

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{1,L,-} + \partial_x(a_L \underline{\mathbf{U}}_{1,L,-}) = 0, & (t,x) \in \Omega_{T,L}^-, \\ \underline{\mathbf{U}}_{1,L,-}|_{t=0} = 0 & . \end{cases}$$

Hence $\underline{\mathbf{U}}_{1,R,+} = 0$ and $\underline{\mathbf{U}}_{1,L,-} = 0$. Actually, we see by induction that for all $n \in \mathbb{N}$, we have $\underline{\mathbf{U}}^{\pm}_{2n+1,R,\pm} = 0$ and $\underline{\mathbf{U}}_{2n+1,L,\pm} = 0$. On the other hand for $n \in \mathbb{N}^*$, the profiles $\underline{\mathbf{U}}_{2n,L,\pm}$ and $\underline{\mathbf{U}}_{2n,R,\pm}$ are given by the following well-posed hyperbolic problems.

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{2n,R,-} + \partial_x(a_R \underline{\mathbf{U}}_{2n,R,-}) = \partial_x^2 \underline{\mathbf{U}}_{2n-2,R,-}, & (t,x) \in \Omega_{T,R}^-, \\ \partial_t \underline{\mathbf{U}}_{2n,L,+} + \partial_x(a_L \underline{\mathbf{U}}_{2n,L,+}) = \partial_x^2 \underline{\mathbf{U}}_{2n-2,L,+}, & (t,x) \in \Omega_{T,L}^+, \\ \begin{pmatrix} \underline{\mathbf{U}}_{2n,L,+}|_{x=0} \\ \underline{\mathbf{U}}_{2n,R,-}|_{x=0} \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ -\left(\partial_x \underline{\mathbf{U}}_{2n-2,R,-}|_{x=0} - \partial_x \underline{\mathbf{U}}_{2n,L,+}|_{x=0}\right) \end{pmatrix} \end{cases}$$

where $M := \begin{pmatrix} 1 & -1 \\ a_L|_{x=0} & -a_R|_{x=0} \end{pmatrix}$; remark that the matrix M is non-singular since $a_L|_{x=0} - a_R|_{x=0} < 0$.

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{2n,R,+} + \partial_x(a_R \underline{\mathbf{U}}_{2n,R,+}) = \partial_x^2 \underline{\mathbf{U}}_{2n-2,R,+}, & (t,x) \in \Omega_{T,R}^+, \\ \underline{\mathbf{U}}_{2n,R,+}|_{t=0} = 0 & . \end{cases}$$

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{2n,L,-} + \partial_x(a_L \underline{\mathbf{U}}_{2n,L,-}) = \partial_x^2 \underline{\mathbf{U}}_{2n-2,L,-}, & (t,x) \in \Omega_{T,L}^-, \\ \underline{\mathbf{U}}_{2n,L,-}|_{t=0} = 0 & . \end{cases}$$

In conclusion, all the profiles $\underline{\mathbf{U}}_n$ are constructed by induction.

We turn now to the construction of the boundary layer profiles $U^c_{L,j,\pm}(t,\theta_L)$ and $U^c_{R,j,\pm}(t,\theta_R)$. We will use the relations imposed on the profiles by the transmission conditions: $[u^\varepsilon_{app}]_{\Gamma_R} = 0$, $[\partial_x u^\varepsilon_{app}]_{\Gamma_R} = 0$, $[u^\varepsilon_{app}]_{\Gamma_L} = 0$, and $[\partial_x u^\varepsilon_{app}]_{\Gamma_L} = 0$; $[u^\varepsilon_{app}]_{\Gamma_R}$ stands for the jump of u^ε_{app} through Γ_R defined, for all $t \in [0,T]$ by:

$$[u^\varepsilon_{app}]_{\Gamma_R}(t) := \lim_{x \to x_R(t), x > x_R(t)} u^\varepsilon_{app}\left(t, x, \frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right) - \lim_{x \to x_R(t), x < x_R(t)} u^\varepsilon_{app}\left(t, x, \frac{\varphi_R(t,x)}{\sqrt{\varepsilon}}\right),$$

where we recall that $x_R(t)$ is the unique x such that $(t,x) \in \Gamma_R$. $[u^\varepsilon_{app}]_{\Gamma_L}(t)$ is defined the same way. Because u^ε_{app} belongs to $C^1((0,T) \times \mathbb{R}^*)$, for all $0 \leq j \leq M$, we have:

$$[U^c_{L,j}]_L = -[\underline{\mathbf{U}}_{L,j}]_{\Gamma_L},$$

$$[U^c_{R,j}]_R = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}.$$

Let $[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}$ be given, for all $t \in (0,T)$, by:

$$[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}(t) = \lim_{x \to x_R(t), x > x_R(t)} \underline{\mathbf{U}}_{R,j,+}(t,x) - \lim_{x \to x_R(t), x < x_R(t)} \underline{\mathbf{U}}_{R,j,-}(t,x)$$

and $[U^c_{R,j}]_R$ be defined, for all $t \in (0,T)$, by:

$$[U^c_{R,j}]_R(t) = \lim_{\theta_R \to 0^+} U^c_{R,j,+}(t,\theta_R) - \lim_{\theta_R \to 0^-} U^c_{R,j,-}(t,\theta_R).$$

To avoid writing the exact symmetric equations on $\{x > 0\}$ and $\{x < 0\}$, let us only proceed with the construction of the boundary layer profiles $U^c_{R,j,\pm}$. Referring to the computations above, for all $1 \leq j \leq M+1$, the following quantity must not have any Dirac measure in it:

$$\partial_x \partial_{\theta_R} U^c_{R,j-1} + \frac{1}{2(\partial_x \varphi)} \partial_x(\partial_x(\underline{\mathbf{U}}_{R,j-2} + U^c_{R,j-2})),$$

Our first boundary condition: $[U^c_{L,j}]_L = -[\underline{\mathbf{U}}_{L,j}]_{\Gamma_L}$, ensures that, even if $\partial_x(\underline{\mathbf{U}}_{R,j-2} + U^c_{R,j-2})$ is, in general, discontinuous on Γ_T, it has no Dirac

Measure. $\partial_x(\partial_x(\underline{\mathbf{U}}_{R,j-2} + U^c_{R,j-2}))$ is the derivative of such a function and thus has a Dirac Measure. Let us describe this singularity: if we fix $t = t_0$, the Dirac measure forming is

$$\left([\partial_x \underline{\mathbf{U}}_{R,j-2}]|_{x=x_R(t_0)} + [\partial_x U^c_{R,j-2}]_R(t_0)\right)\ \delta_{x=x_R(t_0)}.$$

Hence the Dirac measure forming in $\frac{1}{2(\partial_x\varphi)}\partial_x(\partial_x(\underline{\mathbf{U}}_{R,j-2} + U^c_{R,j-2}))$ is

$$\frac{1}{2(\partial_x\varphi)|_{x=x_R(t_0)}}\left([\partial_x \underline{\mathbf{U}}_{R,j-2}(t_0)]|_{x=x_R(t_0)} + [\partial_x U^c_{R,j-2}(t_0)]_R\right)\ \delta_{x=x_R(t_0)}.$$

where $[\omega]|_{x=x_R(t_0)} = \lim_{x\to x_R(t_0), x>x_R(t_0)} \omega - \lim_{x\to x_R(t_0), x<x_R(t_0)} \omega$.
On the other hand, if $\partial_{\theta_R} U^c_{R,j-1}$ is discontinuous through Γ_R, $\partial_x\left(\partial_{\theta_R} U^c_{R,j-1}\right)$ has a Dirac measure given, for $t = t_0$ by:

$$[\partial_{\theta_R} U^c_{R,j-1}]_R\ \delta_{x=x_R(t_0)}.$$

The game is to construct the boundary layer profiles such that the sum of the two Dirac measures cancel. As a result, the second boundary condition we get is that, $\forall t \in (0,T)$:

$$[\partial_{\theta_R} U^c_{R,j-1}]_R(t) = -\frac{1}{2(\partial_x\varphi)|_{x=x_R(t)}}\left([\partial_x \underline{\mathbf{U}}_{R,j-2}]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-2}(t)]_R\right).$$

The profiles $U^c_{R,0,+}$ and $U^c_{R,0,-}$ are solution of the following heat equation:

$$\begin{cases} \partial_t U^c_{R,0,+} - (\partial_x\varphi_R)^2\partial^2_{\theta_R} U^c_{R,0,+} + (\partial_x a_R)U^c_{R,0,+} = 0 \quad t\in(0,T),\ \{\theta_R > 0\}, \\ \partial_t U^c_{R,0,-} - (\partial_x\varphi_R)^2\partial^2_{\theta_R} U^c_{R,0,-} + (\partial_x a_R)U^c_{R,j,-} = 0 \quad t\in(0,T),\ \{\theta_R < 0\}, \\ [U^c_{R,0}]_R(t) = -[u_R]_{\Gamma_R}, \quad \forall t\in(0,T), \\ [\partial_{\theta_R} U^c_{R,j}]_R(t) = 0, \quad \forall t\in(0,T), \\ U^c_{R,j,+}|_{t=0} = 0, \\ U^c_{R,j,-}|_{t=0} = 0 \quad . \end{cases}$$

Note well that, since $[u_R]_{\Gamma_R} \neq 0$, the profiles $U^c_{R,0}$ and $U^c_{L,0}$ are not equal to zero.

For all $1 \leq j \leq M$, the profiles $U^c_{R,j,+}$ and $U^c_{R,j,-}$ are given by:

$$\begin{cases} \partial_t U^c_{R,j,+} - (\partial_x \varphi_R)^2 \partial^2_{\theta_R} U^c_{R,j,+} + (\partial_x a_R) U^c_{R,j,+} = (\partial^2_x \varphi_R) \partial_{\theta_R} U^c_{R,j-1,+} & t \in (0,T),\ \{\theta_R > 0\}, \\ \partial_t U^c_{R,j,-} - (\partial_x \varphi_R)^2 \partial^2_{\theta_R} U^c_{R,j,-} + (\partial_x a_R) U^c_{R,j,-} = (\partial^2_x \varphi_R) \partial_{\theta_R} U^c_{R,j-1,-} & t \in (0,T),\ \{\theta_R < 0\}, \\ [U^c_{R,j}]_R(t) = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}, \quad \forall t \in (0,T), \\ [\partial_{\theta_R} U^c_{R,j}]_R(t) = -\dfrac{1}{2(\partial_x \varphi)|_{x=x_R(t)}} \left([\partial_x \underline{\mathbf{U}}_{R,j-1}(t)]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-1}(t)]_R \right), \quad \forall t \in (0,T), \\ U^c_{R,j,+}|_{t=0} = 0, \\ U^c_{R,j,-}|_{t=0} = 0 \quad . \end{cases}$$

Let us now prove the well-posedness of these problems. We take $\psi_{R,j}$ in $H^\infty((0,T) \times \mathbb{R}^*)$ such that

$$[\psi_{R,j}]_R = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R},$$

and

$$[\partial_{\theta_R} \psi_{R,j}]_R(t) = -\frac{1}{2(\partial_x \varphi)|_{x=x_R(t)}} \left([\partial_x \underline{\mathbf{U}}_{R,j-1}(t)]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-1}(t)]_R \right).$$

We can then compute $U^c_{R,j} := U^c_{R,j,+} \mathbf{1}_{\theta_R > 0} + U^c_{R,j,-} \mathbf{1}_{\theta_R < 0}$ by:

$$U^c_{R,j} := \psi_{R,j} + V^c_{R,j}.$$

$V^c_{R,j}$ is then the solution of the classical heat equation:

$$\begin{cases} \partial_t V^c_{R,j} - (\partial_x \varphi_R)^2 \partial^2_{\theta_R}) V^c_{R,j} + (\partial_x a_R) V^c_{R,j} = \varphi^*_{R,j}, \quad (t, \theta_R) \in (0,T) \times \mathbb{R}, \\ V^c_{R,j}|_{t=0} = 0 \quad . \end{cases}$$

and $\varphi^*_{R,j}$ is given by:

$$\varphi^*_{R,j} := -\left(\partial_t \psi_{R,j} - (\partial_x \varphi_R)^2 \partial^2_{\theta_R} \psi_{R,j} + (\partial_x a_R) \psi_{R,j} \right) + (\partial^2_x \varphi_R) \partial_{\theta_R} U^c_{R,j-1}.$$

The profiles can thus be constructed by induction using the scheme just introduced.

We will now prove stability estimates.
We define the error $w^\varepsilon := u^\varepsilon_{app} - u^\varepsilon$. Let us denote by $w^{\varepsilon\pm}$ the restriction of w^ε to $\pm x > 0$. $(w^{\varepsilon+}, w^{\varepsilon-})$ is then solution of the transmission

problem:

$$\begin{cases} \partial_t w^{\varepsilon+} + \partial_x(a_R w^{\varepsilon+}) - \varepsilon\partial_x^2 w^{\varepsilon+} = \varepsilon^M R^{\varepsilon+}, & x>0, t\in[0,T], \\ \partial_t w^{\varepsilon-} + \partial_x(a_L w^{\varepsilon-}) - \varepsilon\partial_x^2 w^{\varepsilon-} = \varepsilon^M R^{\varepsilon-}, & x<0, t\in[0,T], \\ w^{\varepsilon+}|_{x=0^+} - w^{\varepsilon-}|_{x=0^-} = 0, \\ a_R w^{\varepsilon+}|_{x=0^+} - \varepsilon\partial_x w^{\varepsilon+}|_{x=0^+} = a_L w^{\varepsilon-}|_{x=0^-} - \varepsilon\partial_x w^{\varepsilon-}|_{x=0^-}, \\ w^{\varepsilon+}|_{t=0} = 0, \qquad \forall x>0, \\ w^{\varepsilon-}|_{t=0} = 0, \qquad \forall x<0. \end{cases}$$

By construction of our approximate solution, R^ε belongs to $L^\infty([0,T]:L^2(\mathbb{R}))$. Multiplying by the solution and integrating by parts, we get, for $\{x>0\}$:

$$\frac{1}{2}\frac{d}{dt}\|w^{\varepsilon+}\|^2_{L^2(\mathbb{R}^*_+)} + \varepsilon\|\partial_x w^{\varepsilon+}\|^2_{L^2(\mathbb{R}^*_+)} - \frac{a_R|_{x=0}}{2}\left(w^{\varepsilon+}|_{x=0}\right)^2 + \varepsilon w^{\varepsilon+}\partial_x w^{\varepsilon+}|_{x=0}$$
$$= \varepsilon^M \int_0^\infty R^{\varepsilon+} w^{\varepsilon+}\, dx - 2\int_0^\infty \partial_x a_R (w^{\varepsilon+})^2\, dx.$$

Note that:

$$-\frac{a_R|_{x=0}}{2}\left(w^{\varepsilon+}|_{x=0}\right)^2 + \varepsilon w^{\varepsilon+}\partial_x w^{\varepsilon+}|_{x=0}$$
$$= \frac{a_R|_{x=0}}{2}\left(w^{\varepsilon+}|_{x=0}\right)^2 - w^{\varepsilon+}|_{x=0}\left(a_R w^{\varepsilon+}|_{x=0} - \varepsilon\partial_x w^{\varepsilon+}|_{x=0}\right).$$

And, for $\{x<0\}$, we have:

$$\frac{1}{2}\frac{d}{dt}\|w^{\varepsilon-}\|^2_{L^2(\mathbb{R}^*_-)} + \varepsilon\|\partial_x w^{\varepsilon-}\|^2_{L^2(\mathbb{R}^*_-)} + \frac{a_L|_{x=0}}{2}\left(w^{\varepsilon-}|_{x=0}\right)^2 - \varepsilon w^{\varepsilon-}\partial_x w^{\varepsilon-}|_{x=0}$$
$$= \varepsilon^M \int_{-\infty}^0 R^{\varepsilon-} w^{\varepsilon-}\, dx - 2\int_{-\infty}^0 \partial_x a_L (w^{\varepsilon-})^2\, dx.$$

Note that:

$$\frac{a_L|_{x=0}}{2}\left(w^{\varepsilon-}|_{x=0}\right)^2 - \varepsilon w^{\varepsilon-}\partial_x w^{\varepsilon-}|_{x=0}$$
$$= -\frac{a_L|_{x=0}}{2}\left(w^{\varepsilon-}|_{x=0}\right)^2 + w^{\varepsilon-}|_{x=0}\left(a_L w^{\varepsilon-}|_{x=0} - \varepsilon\partial_x w^{\varepsilon-}|_{x=0}\right).$$

Thanks to our boundary condition, there holds:

$$w^{\varepsilon+}|_{x=0}\left(a_R w^{\varepsilon+}|_{x=0} - \varepsilon\partial_x w^{\varepsilon+}|_{x=0}\right) = w^{\varepsilon-}|_{x=0}\left(a_L w^{\varepsilon-}|_{x=0} - \varepsilon\partial_x w^{\varepsilon-}|_{x=0}\right)$$

Thus, by adding our estimates, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|w^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon\|\partial_x w^\varepsilon\|^2_{L^2(\mathbb{R})} + \frac{a_R|_{x=0} - a_L|_{x=0}}{2}\left(w^\varepsilon|_{x=0}\right)^2$$

$$= \varepsilon^M \int_{-\infty}^{\infty} R^\varepsilon w^\varepsilon \, dx - 2\int_0^\infty \partial_x a_R (w^{\varepsilon +})^2 \, dx - 2\int_{-\infty}^0 \partial_x a_L (w^{\varepsilon -})^2 \, dx.$$

$$\left|\int_{-\infty}^{\infty} R^\varepsilon w^\varepsilon \, dx - 2\int_0^\infty \partial_x a_R (w^{\varepsilon +})^2 \, dx - 2\int_{-\infty}^0 \partial_x a_L (w^{\varepsilon -})^2 \, dx\right| \leq \frac{1}{2}\|R^\varepsilon\|^2_{L^2(\mathbb{R})} + C\|w^\varepsilon\|^2_{L^2(\mathbb{R})},$$

where $C = \frac{1}{2} + 2Max\left(\sup_{(t,x)\in\Omega_L}|\partial_x a_L|, \sup_{(t,x)\in\Omega_R}|\partial_x a_R|\right)$.
Since $a_R|_{x=0} > 0$ and $a_L|_{x=0} < 0$, by Gronwall Lemma, we get the simplified estimate:

$$\|w^\varepsilon\|^2_{L^2(\mathbb{R})}(t) \leq \frac{1}{2}\varepsilon^M \int_0^T e^{C(t-s)}\|R^\varepsilon\|^2_{L^2(\mathbb{R})}(s) \, ds.$$

Constructing the profiles up to order $M = 1$, we get then that there is $c > 0$, independent of ε, such that:

$$\|w^\varepsilon\|^2_{L^\infty([0,T]:L^2(\mathbb{R}))} \leq c\varepsilon,$$

thus achieving our proof.

□

2.3 Treatment of the ingoing case.

Let us now introduce our second result. Our second result concerns the case where, for all $t \in [0, T]$, the coefficient a satisfies:

$$a(t, 0^+) < 0,$$

$$a(t, 0^-) > 0.$$

During the study of a similar problem, Poupaud and Rascle show in [PR97] the formation of a Dirac measure on $\{x = 0\}$ for their solution. We show that a Dirac-measure also forms in the small viscosity limit. We give an asymptotic expansion of the solution u^ε of (2.1.3), which shows explicitely the convergence to the generalized measure-valued solution $\underline{u}$. The main result is stated in Corollary 2.3.3 . The problem we consider here appears as one very simple example of the arising of a 'δ-measure' in the vanishing viscosity limit. Note that, by using viscous approaches as well, Joseph ([Jos93]) and Tan, Zhang , Zheng ([TZZ94]) describe an analogous phenomenon, called δ-shockwave. We will denote by $[\theta]|_{x=0}$ the jump of θ through $\{x = 0\}$ i.e

$$\theta(.,0^+) - \theta(.,0^-).$$

A piecewise smooth u^ε is solution of (2.1.3) iff its restrictions to $\pm x > 0$ satisfies the equation on $\pm x > 0$ and

$$[a(.,x)u^\varepsilon - \varepsilon\partial_x u^\varepsilon]|_{x=0} = 0,$$

which is the corresponding Rankine-Hugoniot condition. The hyperbolic-parabolic problem (2.1.3) reformulates then as the following transmission problem:
(2.3.1)

$$\begin{cases} \partial_t u^{\varepsilon+} + \partial_x(a^+u^{\varepsilon+}) - \varepsilon\partial_x^2 u^{\varepsilon+} = f^+, & \{x>0\}, t\in[0;T], \\ \partial_t u^{\varepsilon-} + \partial_x(a^-u^{\varepsilon-}) - \varepsilon\partial_x^2 u^{\varepsilon-} = f^-, & \{x<0\}, t\in[0;T], \\ u^{\varepsilon+}|_{x=0^+} - u^{\varepsilon-}|_{x=0^-} = 0, & \\ a^+u^{\varepsilon+}|_{x=0^+} - \varepsilon\partial_x u^{\varepsilon+}|_{x=0^+} = a^-u^{\varepsilon-}|_{x=0^-} - \varepsilon\partial_x u^{\varepsilon-}|_{x=0^-}, & \\ u^{\varepsilon+}|_{t=0} = h^+, & \\ u^{\varepsilon-}|_{t=0} = h^- \quad , & \end{cases}$$

with $u^{\varepsilon+} = u^\varepsilon|_{x>0}$, $a^+ = a|_{x>0}$, $f^+ = f|_{x>0}$, $h^+ = h|_{x>0}$ and $u^{\varepsilon-} = u^\varepsilon|_{x<0}$, $a^- = a|_{x<0}$, $f^- = f|_{x<0}$, $h^- = h|_{x<0}$. Problem (2.3.1) can be reformulated as the doubled problem on a half-space:
(2.3.2)

$$\begin{cases} \partial_t\widetilde{u}^\varepsilon + \partial_x(\widetilde{A}\widetilde{u}^\varepsilon) - \varepsilon\partial_x^2\widetilde{u}^\varepsilon = \widetilde{f}(t,x), & \{x>0\}, t\in[0;T], \\ \mathcal{M}_c\widetilde{u}^\varepsilon|_{x=0} = 0, & \\ \widetilde{u}^\varepsilon|_{t=0} = \underline{h} \quad . & \end{cases}$$

Let us precise how problem (2.3.2) is deduced from problem (2.3.1): $\widetilde{u}^\varepsilon$ is a two dimensional vector which first component [resp second component] is $u^{\varepsilon+}(t,x)$ [resp $u^{\varepsilon-}(t,-x)$]. $\widetilde{A}$ is defined by:

$$\widetilde{A}(t,x) = \begin{bmatrix} a^+(t,x) & 0 \\ 0 & -a^-(t,-x) \end{bmatrix},$$

and $\mathcal{M}_c$ is given as follow:

$$\mathcal{M}_c = \left[\begin{array}{cc} 1 & -1 \\ a^+(t,0) - \varepsilon\partial_x & -a^-(t,0) - \varepsilon\partial_x \end{array} \right].$$

In order to prove our main result, there will be two steps: first, we will construct formally an approximate solution of the mixed parabolic problem (2.3.2) then validate it through the adequate energy estimates. Let us detail the form of our approximate solution, $\widetilde{u}^\varepsilon_{app}$ will be constructed as a WKB expansion up to order M of the form:

$$\widetilde{u}^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \sum_{n\geq -1} \varepsilon^n \, \mathbf{U}_n\big(t,x,x/\varepsilon\big), \tag{2.3.3}$$

where $\mathbf{U}_n$ belongs to the space of profiles $\mathcal{P}^*$. Let us define $\mathcal{P}^*$: $\mathbf{U}_n(t,x,z)$ (z is the fast variable x/ε) belongs to $\mathcal{P}^*$ iff it writes:

$$\mathbf{U}_n(t,x,z) = \underline{\mathbf{U}}_n(t,x) + \mathbf{U}^*_n(t,z)$$

with $\underline{\mathbf{U}}_n \in H^\infty([0,T]\times\mathbb{R}^*_+)$ and $\mathbf{U}^*_n(t,z) \in e^{-\delta z}H^\infty([0,T]\times\mathbb{R}^*_+)$ for some $\delta > 0$. In addition, we prescribe $\underline{\mathbf{U}}_{-1}(t,x) = 0$ for obvious reasons. For our treatment, we will see that nonconservative hyperbolic problems are easier to deal with than conservative ones. Moreover, under our assumptions on f and h, a nonconservative hyperbolic problem can be obtained by integrating ours, yielding the desired energy estimates.

We begin by introducing the integrated equation:

$$\begin{cases} \partial_t v^\varepsilon + a(t,x)\partial_x v^\varepsilon - \varepsilon\partial_x^2 v^\varepsilon = F, & (t,x)\in[0,T]\times\mathbb{R}, \\ v^\varepsilon|_{t=0} = H, \end{cases} \tag{2.3.4}$$

where F and H are given by:

$$F = F^+ + F^- := \int_{+\infty}^x f(t,y)dy\mathbf{1}_{x>0} + \int_{-\infty}^x f(t,y)dy\mathbf{1}_{x<0},$$

$$H = H^+ + H^- := \int_{+\infty}^x h(y)dy\mathbf{1}_{x>0} + \int_{-\infty}^x h(y)dy\mathbf{1}_{x<0}.$$

Since f belongs to $C_0^\infty([0,T]\times\mathbb{R})$ and h belongs to $C_0^\infty(\mathbb{R})$, we obtain that $F^\pm$ belongs to $H^\infty([0,T]\times\mathbb{R}^*_\pm)$ and $H^\pm$ belongs to $H^\infty(\mathbb{R}^*_\pm)$. By

[Ike71], for all fixed $\varepsilon > 0$, the parabolic problem (2.3.4) has a unique solution:

$$v^\varepsilon \in C([0,T] : L^2(\mathbb{R})).$$

As a result, the solution u^ε of (2.1.3) satisfies: $u^\varepsilon = \partial_x v^\varepsilon$.

We will now establish Stability estimates for the hyperbolic-parabolic problem (2.1.3). These estimates will be proved by derivation of the stability estimates holding true for (2.3.4). Take C_a given by:

$$C_a := 1 + max(\|\partial_x a^+\|_{L^\infty}, \|\partial_x a^-\|_{L^\infty}).$$

We will now prove the following Proposition:

Proposition 2.3.1. *For all $0 < \varepsilon < 1$ and $t \in [0,T]$:*

$$\int_0^T e^{-C_a t} \|u^\varepsilon\|^2_{L^2(\mathbb{R})} \, dt \leq \frac{1}{2\varepsilon} \left(\|H\|^2_{L^2(\mathbb{R})} + \int_0^T e^{-C_a t} \|F\|^2_{L^2(\mathbb{R})} dt \right)$$

Proof. The proof unfolds in two main steps. In a first step, stability estimates are established for (2.3.4). In a second step, exploiting the fact that the solution of problem (2.1.3) can be obtained by derivation of the solution of problem (2.3.4), stability estimates on (2.1.3) are easily deduced from the stability estimates obtained on (2.3.4). We will rather work on the reformulation of the nonconservative hyperbolic-parabolic problem (2.3.4) as the doubled problem on a half space:

$$\begin{cases} \partial_t \widetilde{v}^\varepsilon + \widetilde{A} \partial_x \widetilde{v}^\varepsilon - \varepsilon \partial_x^2 \widetilde{v}^\varepsilon = \widetilde{F}(t,x), & x > 0, \ t \in [0,T], \\ \mathcal{M}_{nc} \widetilde{v}^\varepsilon|_{x=0} = 0, \\ \widetilde{v}^\varepsilon|_{t=0} = \widetilde{H}, \end{cases} \tag{2.3.5}$$

with, for all $x > 0$ and $t \in [0,T]$:

$$\widetilde{v}^\varepsilon(t,x) = \begin{pmatrix} \widetilde{v}^{\varepsilon +}(t,x) := v^\varepsilon(t,x) \\ \widetilde{v}^{\varepsilon -}(t,x) := v^\varepsilon(t,-x) \end{pmatrix}, \quad \widetilde{F} = \begin{pmatrix} F^+(t,x) \\ F^-(t,-x) \end{pmatrix}, \quad \widetilde{H} = \begin{pmatrix} H^+(t,x) \\ H^-(t,-x) \end{pmatrix},$$

$$\widetilde{A} = \begin{bmatrix} a^+(t,x) & 0 \\ 0 & -a^-(t,-x) \end{bmatrix} \quad \textit{and} \quad \mathcal{M}_{nc} = \begin{bmatrix} 1 & -1 \\ \partial_x & \partial_x \end{bmatrix}.$$

Multiplying (2.3.5) by $\widetilde{v}^\varepsilon$ and integrating with respect to x between 0 and ∞ gives, abbreviating $\|.\|_{L^2(\mathbb{R}_+^*)}$ by $\|.\|_{L^2}$

$$\|\widetilde{v}^\varepsilon\|_{L^2}^2(t) \leq \int_0^t e^{C_a(t-s)}\|\widetilde{F}(s,.)\|_{L^2}^2 ds + e^{C_a t}\|\widetilde{H}\|_{L^2}$$

This gives that $v^\varepsilon \in L^\infty([0,T] : L^2(\mathbb{R}))$ for all finite time $T > 0$.
Moreover,

$$\frac{d}{dt}\|\widetilde{v}^\varepsilon\|_{L^2}^2 + 2\varepsilon\|\partial_x \widetilde{v}^\varepsilon\|_{L^2}^2 \leq \|\widetilde{F}\|_{L^2}^2 + C_a\|\widetilde{v}^\varepsilon\|_{L^2}^2$$

Hence, for all $t \in [0,T]$ and $0 < \varepsilon < 1$:

$$\int_0^T e^{-C_a t}\|\partial_x v^\varepsilon\|_{L^2(\mathbb{R})}^2\, dt \leq \frac{1}{2\varepsilon}\left(\|H\|_{L^2(\mathbb{R})}^2 + \int_0^T e^{-C_a t}\|F\|_{L^2(\mathbb{R})}^2 dt\right)$$

This concludes the proof of Proposition 2.3.1.

□

Let us now construct an approximate solution u_a^ε of equation (2.1.3). We will construct an approximate solution of (2.1.3) at any order, according to ansatz 2.3.3.
For all $-1 \leq n \leq M$, we adopt the following notations:

$$[\mathbf{U}_n^*]_{z=0} := \mathbf{U}_n^{*+}|_{z=0} - \mathbf{U}_n^{*-}|_{z=0},$$

$$[a^{-1}(\partial_t \mathbf{U}_n^*)]_{z=0} := (a^+)^{-1}(\partial_t \mathbf{U}_n^{*+})|_{z=0} - (a^-)^{-1}(\partial_t \mathbf{U}_n^{*-})|_{z=0},$$

$$[\underline{\mathbf{U}}_n]_{x=0} := \underline{\mathbf{U}}_n^+|_{x=0} - \underline{\mathbf{U}}_n^-|_{x=0},$$

$$[\partial_t \underline{\mathbf{U}}_n]_{x=0} := (\partial_t \underline{\mathbf{U}}_n^+)|_{x=0} - (\partial_t \underline{\mathbf{U}}_n^-)|_{x=0},$$

$$[\partial_x \underline{\mathbf{U}}_n]_{x=0} := (\partial_x \underline{\mathbf{U}}_n^+)|_{x=0} + (\partial_x \underline{\mathbf{U}}_n^-)|_{x=0},$$

$$[a\underline{\mathbf{U}}_n]_{x=0} := a^+\underline{\mathbf{U}}_n^+|_{x=0} - a^-\underline{\mathbf{U}}_n^-|_{x=0}.$$

We will compute the $M+1$ first $\mathbf{U}_j^*$ profiles and the $M+2$ first $\underline{U}_j$ profiles. The boundary conditions $\mathcal{M}_c \widetilde{u}_{app}^\varepsilon|_{x=0} = 0$ are translated on the profiles by:

$$\begin{cases} [a\mathbf{U}_n^* - \partial_z \mathbf{U}_n^*]_{z=0} = -[a\underline{\mathbf{U}}_n - \partial_x \underline{\mathbf{U}}_{n-1}]_{x=0}, \\ \mathbf{U}_n^{*+}|_{z=0} - \mathbf{U}_n^{*-}|_{z=0} = -\left(\underline{\mathbf{U}}_n^+|_{x=0} - \underline{\mathbf{U}}_n^-|_{x=0}\right), \end{cases}$$

where $[a\underline{\mathbf{U}}_n - \partial_x \underline{\mathbf{U}}_{n-1}]_{x=0} := a^+ \underline{\mathbf{U}}_n^+|_{x=0} - \partial_x \underline{\mathbf{U}}_{n-1}^+|_{x=0} - \big(a^- \underline{\mathbf{U}}_n^-|_{x=0} + \partial_x \underline{\mathbf{U}}_{n-1}^-|_{x=0}\big)$ and $[a\mathbf{U}_n^* - \partial_z \mathbf{U}_n^*]_{z=0} := a^+ \mathbf{U}_n^{*+}|_{z=0} - \partial_z \mathbf{U}_n^{*+}|_{z=0} - (a^- \mathbf{U}_n^{*-}|_{z=0} + \partial_z \mathbf{U}_n^{*+}|_{z=0})$. Plugging (2.3.3) into the equation (2.3.2) and identifying the terms with same powers in ε gives the following profiles equations: The profiles $\underline{\mathbf{U}}_j$ satisfy

$$\underline{\mathbf{U}}_{-1} = 0,$$

and $\forall 0 \leq n \leq M+1$

$$\begin{cases} \partial_t \underline{\mathbf{U}}_n + \partial_x (\widetilde{A} \underline{\mathbf{U}}_n) = \underline{\boldsymbol{\varphi}}_n, \\ \underline{\mathbf{U}}_n|_{t=0} = 0. \end{cases}$$

Notice that $\underline{\boldsymbol{\varphi}}_0 := \widetilde{f}$ being known, $\underline{\mathbf{U}}_0$ is deduced from it. $\underline{\boldsymbol{\varphi}}_1 := \partial_x^2 \underline{\mathbf{U}}_0$ is then known, which gives $\underline{\mathbf{U}}_1$, and so on. All the profiles $\underline{\mathbf{U}}_j$ having already be computed above, the profiles $\mathbf{U}_j^*$ are deduced from them as solution of the following well-posed equations:

$$\begin{cases} \partial_z^2 \mathbf{U}_{-1}^* - \partial_z (\widetilde{A} \mathbf{U}_{-1}^*) = 0, \\ [\mathbf{U}_{-1}^*]_{z=0} = 0, \\ [a^{-1}(\partial_t \mathbf{U}_{-1}^*)]_{z=0} = [a\underline{\mathbf{U}}_0]_{x=0}, \end{cases}$$

$$\begin{cases} \partial_z^2 \mathbf{U}_0^* - \partial_z (\widetilde{A} \mathbf{U}_0^*) = \partial_t \mathbf{U}_{-1}^*, \\ [\mathbf{U}_0^*]_{z=0} = -[\underline{\mathbf{U}}_0]_{x=0}, \\ [a^{-1}(\partial_t \mathbf{U}_0^*)]_{z=0} = [a\underline{\mathbf{U}}_1]_{x=0} - [\partial_x \underline{\mathbf{U}}_0]_{x=0}, \end{cases}$$

and, for all $1 \leq n \leq M$, we have:

$$\begin{cases} \partial_z^2 \mathbf{U}_n^* - \partial_z (\widetilde{A} \mathbf{U}_n^*) = \partial_t \mathbf{U}_{n-1}^*, \\ [\mathbf{U}_n^*]_{z=0} = -[\underline{\mathbf{U}}_n]_{x=0}, \\ [a^{-1}(\partial_t \mathbf{U}_n^*)]_{z=0} = [a\underline{\mathbf{U}}_{n+1}]_{x=0} - [\partial_x \underline{\mathbf{U}}_n]_{x=0}. \end{cases}$$

To sum up, we have constructed $\widetilde{u}_{app}^\varepsilon$ as a finite expansion of the form 2.3.3 satisfying:

$$\begin{cases} \partial_t \widetilde{u}_{app}^\varepsilon + \partial_x (\widetilde{A} \widetilde{u}_{app}^\varepsilon) - \varepsilon \partial_x^2 \widetilde{u}_{app}^\varepsilon = \widetilde{f}(t,x) + \varepsilon^M R^\varepsilon, \ (t,x) \in [0;T] \times \mathbb{R}_+^*, \\ \mathcal{M}_c \widetilde{u}_{app}^\varepsilon|_{x=0} = 0, \\ \widetilde{u}_{app}^\varepsilon|_{t=0} = \underline{h} \quad , \end{cases}$$

where $\varepsilon^M R^\varepsilon$ is the error we have generated, substituting $\widetilde{u}^\varepsilon$ by $\widetilde{u}^\varepsilon_{app}$.
Let us denote

$$u_a^\varepsilon\left(t,x,\frac{x}{\varepsilon}\right) = \varepsilon^{-1}\mathbf{U}^*_{-1}\left(t,\frac{x}{\varepsilon}\right) + \mathbf{U}^*_0\left(t,\frac{x}{\varepsilon}\right) + \underline{\mathbf{U}}_0(t,x).$$

This is an approximate solution for $M = 1$.

Theorem 2.3.2. *Assume that* $f \in C_0^\infty([0,T]\times\mathbb{R})$ *and* $h \in C_0^\infty(\mathbb{R})$, *then there is a constant* $C > 0$, *such that, for all* $0 < \varepsilon < 1$*:*

$$\int_0^T e^{-C_a t}\|u^\varepsilon - u_a^\varepsilon\|^2_{L^2(\mathbb{R})}\,dt \leq C\varepsilon.$$

Proof. We denote by $w^{\varepsilon\pm}(t,x) = u^{\varepsilon\pm}_{app}(t,\pm x) - u^{\varepsilon\pm}(t,\pm x)$. By linearity, $w^{\varepsilon\pm}$ satisfies the equation:

$$\begin{cases} \partial_t w^{\varepsilon\pm} + a^\pm\partial_x w^{\varepsilon\pm} - \varepsilon\partial_x^2 w^{\varepsilon\pm} = \varepsilon^M R^{\varepsilon\pm}, \quad \{\pm x > 0\}, t \in [0;T] \\ w^{\varepsilon+}|_{x=0} - w^{\varepsilon-}|_{x=0} = 0, \\ \left(a^+ w^{\varepsilon+} - \varepsilon\partial_x w^{\varepsilon+}\right)|_{x=0^+} - \left(a^- w^{\varepsilon-} - \varepsilon\partial_x w^{\varepsilon-}\right)|_{x=0^-} = 0, \\ w^{\varepsilon\pm}|_{t=0} = 0 \quad . \end{cases}$$

We denote $I(R^\varepsilon) := \int_{-\infty}^x R^\varepsilon(t,y)dy\mathbf{1}_{x<0} + \int_\infty^x R^\varepsilon(t,y)dy\mathbf{1}_{x>0}$. We can perform the construction of an approximate solution whose restriction to
$\pm x > 0$ belongs to $H^\infty([0,T]\times\mathbb{R}^*_\pm)$. $I(R^\varepsilon)$ is a linear combination of the profiles involved in this construction thus belonging to $H^\infty([0,T]\times\mathbb{R}^*)$. As a consequence of Proposition 2.3.1 , there holds:

$$\int_0^T e^{-C_a t}\|w^\varepsilon\|^2_{L^2(\mathbb{R})}\,dt \leq \frac{1}{2}\varepsilon^{2M-1}\int_0^T e^{-C_a t}\|I(R^\varepsilon)\|^2_{L^2(\mathbb{R})}dt,$$

which achieves our proof. □

As a Corollary, we obtain the limit of u^ε. Let us note u_0 the function defined by:

$$u_0(t,x) := \underline{\mathbf{U}}_0^+(t,x)\mathbf{1}_{x>0} + \underline{\mathbf{U}}_0^-(t,-x)\mathbf{1}_{x<0},$$

and u_{-1} the function defined by:

$$u_{-1}(t,z) := \mathbf{U}^{*+}_{-1}(t,z)\mathbf{1}_{z\geq 0} + \mathbf{U}^{*+}_{-1}(t,-z)\mathbf{1}_{z<0}.$$

Note that u_{-1} is continuous across $\{z = 0\}$.

Corollary 2.3.3. *When ε tends to zero, u^ε converges in $\mathcal{D}'((0,T)\times\mathbb{R})$ towards $\underline{u}$ which is a measure of the form*

$$\underline{u}(t,.) = C(t)\ \delta_{x=0} + u_0(t,.),$$

where $u_0(t,.)$ is the regular part of the measure, and $C(t)\ \delta_{x=0}$ is the singular part. The function $C(t)$ is

$$C(t) = \int_{\mathbb{R}} u_{-1}(t,y)\ dy.$$

We observe that $\lim_{\varepsilon\to 0^+} \|u^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} = \infty$, and thus there is no constant $C > 0$ such that:

$$\|u^\varepsilon\|_{L^2([0,T]\times\mathbb{R})} \leq C\left(\|f\|_{L^2([0,T]\times\mathbb{R})} + \|h\|_{L^2([0,T]\times\mathbb{R})}\right), \quad \forall\varepsilon > 0.$$

As a consequence, our parabolic problem does not satisfy the Uniform Evans Condition (if it was the case, uniform L^2 estimates in ε would hold).

Chapter 3

Approche Visqueuse pour des Problèmes Linéaires Hyperboliques Non-conservatifs avec des Coefficients Discontinus.

Ce chapitre reprend le papier [For07d] intitulé "Viscous approach for Linear Hyperbolic Systems with Discontinuous Coefficients" soumis à publication en septembre 2007.

Abstract

In this paper, two main results are proved. We consider a nonconservative linear Cauchy problem with discontinuous coefficients accross a noncharacteristic hypersurface. The considered problems need not be the linearization of a shock-wave on a shock front. We introduce then a viscous perturbation of the problem; the viscous solution u^ε depends of the small positive parameter ε. This problem, obtained by small viscous perturbation, is parabolic for fixed positive ε. We prove then, under stability assumptions, the convergence, when $\varepsilon \to 0^+$, of u^ε towards the solution of a well-posed limit hyperbolic problem. Our first result is obtained, in the multi-D framework, for piecewise smooth coefficients and states the convergence of u^ε towards an unique solution. Our second result is the analogous of a result we have proved in [For07c] in the conservative framework. It shows that, in the expansive nonconservative scalar case, that is to say for $sign(xa(x)) > 0$,

our viscous approach successfully singles out a solution. Even for scalar, piecewise constant 1-D nonconservative hyperbolic equations, this result is new and not treated during our analysis performed on systems. For both results, an asymptotic analysis of the convergence is performed at any order, containing a boundary layer analysis. Under our assumptions made for systems, only strong amplitude noncharacteristic boundary layers can form, and are localized on the zone of discontinuity of the coefficient, whereas, in our scalar expansive case, only some weak amplitude characteristic boundary layers can form along some characteristic curves.

3.1 Introduction.

Let us consider a linear hyperbolic system of the form:

$$(3.1.1) \qquad \begin{cases} \partial_t u + \sum_{j=1}^{d} A_j(t,y,x)\partial_j u = f, & (t,y,x) \in \Omega \\ u|_{t=0} = h \quad , \end{cases}$$

where $\Omega = \{(t,y,x) \in (0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}\}$, with $T > 0$ fixed once for all. The unknown $u(t,y,x)$ belongs to $\mathbb{R}^N$ and the matrices A_j are valued in the set of $N \times N$ matrices with real coefficients $\mathcal{M}_N(\mathbb{R})$. Due to the discontinuity of the coefficients, the solution u is, in general, awaited to be discontinuous through $\{x = 0\}$. In such case, $\partial_x u$ has a Dirac measure supported on the hypersurface $\{x = 0\}$. Hence, if the coefficient of the normal derivative A_d is also discontinuous through $\{x = 0\}$, the nonconservative product $A_d \partial_x u$ cease to be well-defined in the sense of distributions; weak solutions for the considered problem thus cannot be defined in a classical way.

The definition of such nonconservative product is of course crucial for defining a notion of weak solutions for such problems. It is an interesting question by itself, solved for instance in a quasi-linear framework by Dal Maso, LeFloch and Murat in [DMLM95] and by LeFloch and Tzavaras in [LT99]. Existence and stability results in a neighboring framework of ours have been obtained by LeFloch ([LeF90]) in a 1-D scalar case and by Crasta and LeFloch ([CL02]) for 1-D systems. The equations studied in [LeF90] and [CL02] can be viewed as linear nonconservative problems with discontinuous coefficients; in these works the discontinuity of the coefficient is linked with a shockwave. Adopting a viscous approach will allow us to avoid the difficult question of giving a sense to the nonconservative product in the linear framework.

The problematic investigated in this paper relates to many scalar works on analogous conservative problems. We can for instance refer to the works of Bouchut, James and Mancini in [BJ98], [BJM05]; by Poupaud and Rascle in [PR97] or by Diperna and Lions in [DL89]. We can also refer to [For07c] by Fornet. The common idea is that another notion of solution has to be introduced to deal with linear hyperbolic

Cauchy problems with discontinuous coefficients. Note that almost all the papers cited before use a different approach to deal with the problem. Like in [For07c] and [For07a], we will opt for a small viscosity approach.

Let us now describe the first result obtained in this paper. We consider the following viscous hyperbolic-parabolic problem:

$$\begin{cases} \mathcal{H}^\varepsilon u^\varepsilon = f, & (t,y,x) \in \Omega, \\ u^\varepsilon|_{t<0} = 0, \end{cases} \tag{3.1.2}$$

where $\mathcal{H}^\varepsilon := \partial_t + \sum_{j=1}^{d-1} A_j \partial_j + A_d \partial_x - \varepsilon \sum_{1 \leq j,k \leq d} \partial_j (B_{j,k} \partial_k \cdot)$, and the coefficients A_j, with $1 \leq j \leq d$, are piecewise smooth and constant outside a compact set. We assume that the discontinuity of the coefficients occurs only through the hypersurface $\{x = 0\}$. The unknown $u^\varepsilon(t,y,x) \in \mathbb{R}^N$, the source term f belongs to $H^\infty((0,T) \times \mathbb{R}^d)$, and satisfies $f|_{t<0} = 0$; this assumption allows to bypass the analysis of the compatibility conditions. In this problem, ε, commonly called viscosity, stands for a small positive parameter. We stress that, if we suppress the terms in $-\varepsilon \partial_x^2$ from our differential operator, the obtained hyperbolic problem has no obvious sense.

We make the classical hyperbolicity and hyperbolicity-parabolicity assumptions, plus we assume the boundary is noncharacteristic. Additionally, we make a transversality assumption and an assumption concerning the sign of the eigenvalues of A_d on each side of $\{x = 0\}$. Last, we suppose a spectral stability condition, which is a Uniform Evans Condition for a related problem, is satisfied.

Under these assumptions, we prove that, when $\varepsilon \to 0^+$, u^ε converges towards u in $L^2((0,T) \times \mathbb{R}^d)$, where $u := u^+ \mathbf{1}_{x \geq 0} + u^- \mathbf{1}_{x<0}$ is solution of a transmission problem of the form:

$$\begin{cases} \partial_t u^+ + \sum_{j=1}^{d} A_j^+ \partial_j u^+ = f^+, & (t,y,x) \in \Omega^+ \\ \partial_t u^- + \sum_{j=1}^{d} A_j^- \partial_j u^- = f^-, & (t,y,x) \in \Omega^- \\ u^+|_{x=0} - u^-|_{x=0} \in \Sigma, \\ u^+|_{t<0} = 0, \quad u^-|_{t<0} = 0 & . \end{cases}$$

where Σ is a linear subspace depending of the choice of the viscosity tensor $\sum_{1 \leq j,k \leq d} \partial_j (B_{j,k} \partial_k .)$; $\Omega^\pm$ denotes $\Omega \bigcap \{\pm x > 0\}$ and the $\pm$ superscripts are used to indicate the restrictions of the concerned functions to $\Omega^\pm$.

A crucial remark is that, for fixed positive ε, (3.1.2) can be put on the form of a parabolic problem on the half-space $\{x > 0\}$ with boundary conditions on $\{x = 0\}$ satisfying a **Uniform Evans Condition**. Moreover, the solution of this parabolic problem on a half-space tends, when ε goes to zero, towards the solution of a mixed hyperbolic problem, defined on $\{x > 0\}$, satisfying a **Uniform Lopatinski Condition**. An analogous theorem, in the nonlinear framework and for a shockwave solution, was proved by Rousset ([Rou03]).

For our first result, with conciseness in mind, the proof of stability is exposed only for 1-D systems with piecewise constant coefficients and the artificial viscosity tensor $B = Id$. The goal is to check that the methods introduced in [Mét04] does apply to our boundary conditions. During this proof, accent is placed on the role played by the Uniform Evans Condition in the proof of our stability estimates via Kreiss-type Symmetrizers.

Let us now expose our second result, which concerns the sense to give to the solution of:

$$\begin{cases} \partial_t u + a(x) \partial_x u = f, & (t,x) \in (0,T) \times \mathbb{R}, \\ u|_{t=0} = h, & . \end{cases}$$

in the case where $a(x) = a^+ \mathbf{1}_{x>0} + a^- \mathbf{1}_{x<0}$, where a^+ is a positive constant and a^- is a negative constant. The source term f belongs to

$C_0^\infty((0,T)\times\mathbb{R})$ and the Cauchy data h belongs to $C_0^\infty(\mathbb{R})$. We assume that the coefficient is piecewise constant in order to simplify the proof of our stability estimates, which uses Kreiss-type symmetrizers. Referring to the sign of the coefficient on each side of $\{x=0\}$, we call such discontinuity of the coefficient expansive. Note that such expansive case was excluded from our previous study on systems by our assumptions. An important point is that, compared to the cases studied for our first result, the expansive case has a quite different qualitative behavior. Indeed, for scalar equations, small amplitude characteristic boundary layers only form in the expansive case.

Our second result states the convergence in the vanishing viscosity limit and in $L^2((0,T)\times\mathbb{R})$ of u^ε, which is solution of:

$$\begin{cases} \partial_t u^\varepsilon + a(x)\partial_x u^\varepsilon - \varepsilon\partial_x^2 u^\varepsilon = f, & (t,x)\in(0,T)\times\mathbb{R}, \\ u^\varepsilon|_{t=0} = h & . \end{cases}$$

towards $\underline{u}\in L^2((0,T)\times\mathbb{R})$, where $\underline{u} := \underline{u}^+\mathbf{1}_{x\geq 0} + \underline{u}^-\mathbf{1}_{x<0}$ is the unique solution of the well-posed, even though **not classical**, transmission problem:

$$\begin{cases} \partial_t\underline{u}^+ + a^+\partial_x\underline{u}^+ = f^+, & (t,x)\in(0,T)\times\mathbb{R}_+^*, \\ \partial_t\underline{u}^- + a^-\partial_x\underline{u}^- = f^-, & (t,x)\in(0,T)\times\mathbb{R}_-^*, \\ \underline{u}^+|_{x=0} - \underline{u}^-|_{x=0} = 0, \\ \partial_x\underline{u}^+|_{x=0} - \partial_x\underline{u}^-|_{x=0} = 0, \\ \underline{u}^+|_{t=0} = h^+, \quad \underline{u}^-|_{t=0} = h^- & . \end{cases}$$

Naturally, $\underline{u}$ is then what could be called the small viscosity solution of (2.1.1). The result seems to be completely new, since the main difficulty was to 'select' a solution among all possible weak solutions. Remark that, this time, by performing explicit computations of the Evans function, we prove that the Uniform Evans Condition holds for our problem thus yielding the desired stability estimates.

3.2 Some results for multi-D nonconservative systems with 'no expansive modes'.

3.2.1 Description of the problem.

We first expose our full set of assumptions for the problem involved in our first result.
We note $y := (x_1, \ldots, x_{d-1})$ and $x := x_d$ and consider the viscous equation:

$$\begin{cases} \mathcal{H}^\varepsilon u^\varepsilon = f, & (t,y,x) \in \Omega, \\ u^\varepsilon|_{t<0} = 0, \end{cases} \tag{3.2.1}$$

where $\mathcal{H}^\varepsilon := \partial_t + \sum_{j=1}^{d-1} A_j \partial_j + A_d \partial_x - \varepsilon \sum_{1 \leq j,k \leq d} \partial_j (B_{j,k} \partial_k .)$, the unknown
$u^\varepsilon(t,y,x) \in \mathbb{R}^N$, the source term f belongs to $H^\infty((0,T) \times \mathbb{R}^d)$, and satisfies $f|_{t<0} = 0$. All the matrices A_j, $1 \leq j \leq d$ are assumed smooth in (t,y,x) on $\pm x > 0$, discontinuous through $\{x = 0\}$ and constant outside a compact set. The matrices $B_{j,k}$ also depends smoothly of (t,y,x) and are constant outside a compact set. We will denote by $A_d^\pm$ the restriction of A_d to $\{\pm x > 0\}$. We assume that the boundary is noncharacteristic:

Assumption 3.2.1 (Noncharacteristic boundary)**.**
$A_d|_{x=0^+}$ and $A_d|_{x=0^-}$ are two nonsingular $N \times N$ matrices with real coefficients.

Moreover, we make the following structure assumption on the discontinuity of A_d through $\{x = 0\}$:

Assumption 3.2.2 (Sign Assumption)**.**

- *The eigenvalues of $A_d^-(t,y,0)$, sorted by increasing order are denoted by $(\lambda_i^-(t,y))_{1 \leq i \leq N}$, and are such that $\lambda_p^- < 0$ and $\lambda_{p+1}^- > 0$.*
- *The eigenvalues of $A_d^+(t,y,0)$, sorted by increasing order are denoted by $(\lambda_i^+(t,y))_{1 \leq i \leq N}$, and satisfy $\lambda_{p+q}^+ < 0$ and $\lambda_{p+q+1}^+ > 0$, with $q \geq 0$.*

We make the following hyperbolicity assumption on the operator

$$\mathcal{H} := \partial_t + \sum_{j=1}^{d} A_j \partial_j :$$

Assumption 3.2.3 (Hyperbolicity with constant multiplicity)**.**

For all $(t, y, x) \in (0, T) \times \mathbb{R}^{d-1} \times \mathbb{R}^*$ *and* $(\eta, \xi) \neq 0_{\mathbb{R}^d}$,

$$\sum_{j=1}^{d-1} \eta_j A_j(t, y, x) + \xi A_d(t, y, x)$$

remains diagonalizable. Moreover, its eigenvalues keep constant multiplicities.

Let us now introduce the symbol of the parabolic part, B, defined by:

$$B(t, y, x, \eta, \xi) := \sum_{j,k<d} \eta_j \eta_k B_{j,k}(t, y, x)$$
$$+ \sum_{j<d} \xi \eta_j (B_{j,d}(t, y, x) + B_{d,j}(t, y, x)) + \xi^2 B_{d,d}(t, y, x).$$

We make then the following hyperbolicity-parabolicity assumption:

Assumption 3.2.4 (Hyperbolicity-Parabolicity)**.**
There is $c > 0$ *such that for all* $(t, y, x) \in (0, T) \times \mathbb{R}^{d-1} \times \mathbb{R}^*$ *and* $(\eta, \xi) \in \mathbb{R}^d$, *the eigenvalues of*

$$i \left(\sum_{j=1}^{d-1} \eta_j A_j(t, y, x) + \xi A_d(t, y, x) \right) + B(t, y, x, \eta, \xi)$$

satisfy $\Re e\ \mu \geq c(|\eta|^2 + \xi^2)$.

In what follows, $\eta := (\eta_1, \dots, \eta_{d-1})$ will denote the Fourier variable dual to y and ξ the Fourier variable dual to x. Let us now introduce some notations in view of writing the Uniform Evans Condition. $\mathbb{A}^{\pm}$ denotes the matrices of $\mathcal{M}_{2N}(\mathbb{C})$ defined by:

$$\mathbb{A}^{\pm}(t, y, x; \zeta) = \begin{pmatrix} 0 & Id \\ \mathcal{M}^{\pm}(t, y, x; \zeta) & \mathcal{A}^{\pm}(t, y, x; \eta) \end{pmatrix},$$

where $\zeta := (\tau, \gamma, \eta)$,

$$\mathcal{M}^{\pm}(t, y, x; \zeta) = B_{d,d}^{-1} A_d^{\pm}(t, y, x) A^{\pm}(t, y, x; \zeta) + B_{d,d}^{-1}(t, y, x) \sum_{j,k=1}^{d-1} \eta_j \eta_k B_{j,k}(t, y, x),$$

with $A^{\pm}$ standing for the symbol of the hyperbolic part defined by:

$$A^{\pm}(t, y, x; \zeta) := (A_d^{\pm})^{-1}(t, y) \left((i\tau + \gamma) Id + \sum_{j=1}^{d-1} i\eta_j A_j(t, y, x) \right).$$

and

$$\mathcal{A}^{\pm}(t, y, x; \eta) = B_{d,d}^{-1} A_d^{\pm}(t, y, x) - B_{d,d}^{-1}(t, y, x) \sum_{j=1}^{d-1} i\eta_j \left(B_{j,d}(t, y, x) + B_{d,j}(t, y, x) \right).$$

We introduce the weight $\Lambda(\zeta)$ used to deal with high frequencies:

$$\Lambda(\zeta) = \left(1 + \tau^2 + \gamma^2 + |\eta|^4\right)^{\frac{1}{4}}.$$

Let J_Λ be the mapping from $\mathbb{C}^N \times \mathbb{C}^N$ to $\mathbb{C}^N \times \mathbb{C}^N$ given by

$$(u, v) \mapsto (u, \Lambda^{-1} v).$$

The scaled negative and positive spaces of the matrices $\mathbb{A}^{\pm}(t, y, x; \eta)$ are defined by:

$$\widetilde{\mathbb{E}}_{\pm}(\mathbb{A}^{\pm}) := J_\Lambda \mathbb{E}_{\pm}(\mathbb{A}^{\pm}).$$

If $\mathbb{E}$ and $\mathbb{F}$ are two linear subspaces of $\mathbb{C}^{2N}$ such that $\dim \mathbb{E} + \dim \mathbb{F} = 2N$, then $det(\mathbb{E}, \mathbb{F})$ stands for the determinant obtained by taking two direct orthonormal bases of $\mathbb{E}$ and $\mathbb{F}$. Our stability assumption writes then:

Assumption 3.2.5 (Uniform Evans Condition).
We assume that $(\widetilde{\mathcal{H}}^{\varepsilon}, \Gamma)$ satisfies the Uniform Evans Condition that is to say that, for all $(t, y) \in (0, T) \times \mathbb{R}^{d-1}$ and $\zeta = (\tau, \eta, \gamma) \in \mathbb{R}^d \times \mathbb{R}^+ - \{0_{\mathbb{R}^{d+1}}\}$, there holds:

$$\widetilde{D}(t, y, \zeta) = \left| det\left(\widetilde{\mathbb{E}}_{-}(\mathbb{A}^{+}(t, y, 0; \zeta)), \widetilde{\mathbb{E}}_{+}(\mathbb{A}^{-}(t, y, 0; \zeta)) \right) \right| \geq C > 0.$$

$\widetilde{D}$ is called the scaled Evans function. The zeros of $\widetilde{D}$ track down the instabilities of our problem.

Assumption 3.2.6 (Transversality)**.**
$\mathbb{E}_-(G_d^+|_{x=0})$ and $\mathbb{E}_+(G_d^-|_{x=0})$ intersects transversally in $\mathbb{R}^N$, which means that:

$$\mathbb{E}_-(G_d^+|_{x=0}) + \mathbb{E}_+(G_d^-|_{x=0}) = \mathbb{R}^N.$$

Let G_d denote the matrix $G_d(t, y, x) := B_{d,d}^{-1} A_d(t, y, x)$. We have then the following Lemma:

Lemma 3.2.7. *$B_{d,d}$ is nonsingular and its eigenvalues satisfy $\Re e\mu \geq c > 0$. Moreover, $G_d|_{x=0^+}$ and $G_d|_{x=0^-}$ have no eigenvalue on the imaginary axis, furthermore*

$$\dim \mathbb{E}_\pm(G_d|_{x=0^+}) = \dim \mathbb{E}_\pm(A_d^+|_{x=0})$$

and

$$\dim \mathbb{E}_\pm(G_d|_{x=0^-}) = \dim \mathbb{E}_\pm(A_d^-|_{x=0}).$$

Proof. This lemma is a consequence of the hyperbolicity-parabolicity assumption. Fixing $\eta = 0$ and $\xi = \xi_0 \neq 0$ in the hyperbolicity-parabolicity assumption gives that the eigenvalues of: $\xi_0^2 B_{d,d} + i\xi_0 A_d$ satisfy $\Re e\mu \geq c\xi_0^2$, for some $c > 0$. Hence the eigenvalues of $B_{d,d} + \frac{i}{\xi_0} A_d$ are such that $\Re e\mu \geq c$. Making ξ_0 tends to wards infinity, we check that $B_{d,d}$ is nonsingular and that its eigenvalues does not come near the imaginary axis. For all $\xi_0 \neq 0$ and $t \in [0, 1]$, the eigenvalues of $tB_{d,d} + (1-t)Id + \frac{i}{\xi_0} A_d$ are such that $\Re e\mu > 0$. Thus $\left(tB_{d,d} + (1-t)Id + \frac{i}{\xi_0} A_d\right)^{-1} A_d$ has no eigenvalue on the imaginary axis. Indeed, if it was the case, it would mean that, for some $\xi_0' \neq 0$, $tB_{d,d} + (1-t)Id + \frac{i}{\xi_0'} A_d$ has also an eigenvalue on the imaginary axis. Since the eigenvalues of $\left(tB_{d,d} + (1-t)Id + \frac{i}{\xi_0} A_d\right)^{-1} A_d$ do not cross the imaginary axis, making ξ_0 tends to infinity and considering in succession $t = 0$ and $t = 1$, we have then proved that G_d has the same number of eigenvalues with positive [resp negative] real part than A_d. In particular, we get that $\dim \mathbb{E}_\pm(G_d|_{x=0^+}) = \dim \mathbb{E}_\pm(A_d^+|_{x=0})$ and $\dim \mathbb{E}_\pm(G_d|_{x=0^-}) = \dim \mathbb{E}_\pm(A_d^-|_{x=0})$. $\square$

3.2.2 Construction of an approximate solution.

We will begin by reformulating the problem (3.2.1). This viscous problem can be recast as a 'doubled' problem on a half space. Let the '+' [resp '-'] superscript denote the restriction of the concerned function to $\{x > 0\}$ [resp $\{x < 0\}$]. We begin by introducing

$$\widetilde{u}^{\varepsilon}(t, y, x) = \begin{pmatrix} u^{\varepsilon +}(t, y, x) \\ u^{\varepsilon -}(t, y, -x) \end{pmatrix},$$

the new source term writes $\widetilde{f}(t, y, x) = \begin{pmatrix} f^{+}(t, y, x) \\ f^{-}(t, y, -x) \end{pmatrix}$, and the new Cauchy data is $\widetilde{h} = \begin{pmatrix} h^{+}(t, y, x) \\ h^{-}(t, y, -x) \end{pmatrix}$, the normal coefficient becomes:

$$\widetilde{A}_d(t, y, x) = \begin{pmatrix} A_d^{+}(t, y, x) & 0 \\ 0 & -A_d^{-}(t, y, -x) \end{pmatrix}$$

We define then the tangential symbol $\widetilde{A}$ as follows:

$$\widetilde{A}(t, y, x; \zeta) = \begin{pmatrix} A^{+}(t, y, x; \zeta) & 0 \\ 0 & A^{-}(t, y, -x; \zeta) \end{pmatrix}.$$

For $1 \leq j \leq d-1$, we denote:

$$\widetilde{A}_j(t, y, x) = \begin{pmatrix} A_j^{+}(t, y, x) & 0 \\ 0 & A_j^{-}(t, y, -x) \end{pmatrix}.$$

Moreover, if both $j \neq d, k \neq d$ or if $j = k = d$, we note:

$$\widetilde{B}_{j,k}(t, y, x) = \begin{pmatrix} B_{j,k}^{+}(t, y, x) & 0 \\ 0 & B_{j,k}^{-}(t, y, -x) \end{pmatrix};$$

and, if $(j = d, k \neq d)$ or $(j \neq d, k = d)$, we write:

$$\widetilde{B}_{j,k}(t, y, x) = \begin{pmatrix} B_{j,k}^{+}(t, y, x) & 0 \\ 0 & -B_{j,k}^{-}(t, y, -x) \end{pmatrix}.$$

Finally, the new boundary condition is:

$$\widetilde{\Gamma} = \begin{pmatrix} Id & -Id \\ \partial_x & \partial_x \end{pmatrix},$$

we obtain then the following equivalent reformulation of the hyperbolic-parabolic viscous problem (3.2.1):

$$
\begin{cases}
\widetilde{\mathcal{H}}^\varepsilon \widetilde{u}^\varepsilon = \widetilde{f}, \quad \{x > 0\}, \\
\widetilde{\Gamma} \widetilde{u}^\varepsilon|_{x=0} = 0, \\
\widetilde{u}^\varepsilon|_{t<0} = 0.
\end{cases} \tag{3.2.2}
$$

where

$$\widetilde{\mathcal{H}}^\varepsilon := \partial_t + \sum_{j=1}^{d-1} \widetilde{A}_j \partial_j + \widetilde{A}_d \partial_x - \varepsilon \sum_{1 \leq j,k \leq N} \partial_j (\widetilde{B}_{j,k} \partial_k .);$$

we will also note

$$\widetilde{\mathcal{H}} := \partial_t + \sum_{j=1}^{d-1} \widetilde{A}_j \partial_j + \widetilde{A}_d \partial_x .$$

We construct an approximate solution of equation (3.2.2) along the following ansatz:

$$\widetilde{u}^\varepsilon_{app}(t,y,x) := \sum_{n=1}^{M} U_n \left(t, y, x, \frac{x}{\varepsilon} \right) \varepsilon^n, \tag{3.2.3}$$

$$U_n(t,y,x,z) := \underline{U}_n(t,y,x) + U_n^*(t,y,x,z),$$

with $\underline{U}_n \in H^\infty((0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}_+^*)$ and $U_n^* \in e^{-\delta z} H^\infty((0,T) \times \mathbb{R}^{d-1} \times \mathbb{R}_+^* \times \mathbb{R}_+^*)$, for some $\delta > 0$. Note that, due to our previous change of unknowns, we have $\underline{U}_n(t,y,x) \in \mathbb{R}^{2N}$ and $U_n^*(t,y,x,z) \in \mathbb{R}^{2N}$. Moreover, we will note:

$$\underline{U}_n(t,y,x) = \begin{pmatrix} \underline{U}_n^+(t,y,x) \\ \underline{U}_n^-(t,y,x) \end{pmatrix}, \qquad U_n^*(t,y,x,z) = \begin{pmatrix} U_n^{*+}(t,y,x,z) \\ U_n^{*-}(t,y,x,z) \end{pmatrix}.$$

Plugging our asymptotic expansion (3.2.3) into the doubled problem (3.2.2), we get the following profiles equations: to begin with, U_0^* satisfies the following ODE in z :

$$
\begin{cases}
\widetilde{A}_d(t,y,x) \partial_z U_0^* - \widetilde{B}_{d,d}(t,y,x) \partial_z^2 U_0^* = 0, \\
U_0^{*+}|_{(z,x)=0} - U_0^{*-}|_{(z,x)=0} = - \left(\underline{U}_0^+|_{x=0} - \underline{U}_0^-|_{x=0} \right), \\
\partial_z U_0^{*+}|_{(z,x)=0} + \partial_z U_0^{*-}|_{(z,x)=0} = 0.
\end{cases}
$$

Denote $\widetilde{G}_d = \widetilde{B}_{d,d}^{-1}\widetilde{A}_d$, the profile U_0^* writes then:

$$U_0^*(t,y,x,z) = e^{\widetilde{G}_d(t,y,x)z}U_0^*(t,y,x,0).$$

Going back to the transmission conditions satisfied by U_0^*, we obtain that $U_0^*|_{(z,x)=0}$ satisfies the relations:

$$\begin{cases} U_0^{*+}|_{(z,x)=0} - U_0^{*-}|_{(z,x)=0} = -\sigma_0(t,y), \\ G_d^+(t,y,0)U_0^{*+}|_{(z,x)=0} - G_d^-(t,y,0)U_0^{*-}|_{(z,x)=0} = 0, \\ U_0^{*+}|_{(z,x)=0} \in \mathbb{E}_-\left(G_d^+(t,y,0)\right), \\ U_0^{*-}|_{(z,x)=0} \in \mathbb{E}_+\left(G_d^-(t,y,0)\right), \end{cases}$$

where $\sigma_0 := \underline{U_0^+}|_{x=0} - \underline{U_0^-}|_{x=0}$, and $G_d^\pm := B_{d,d}^{-1}A_d^\pm$. This algebraic problem is well-posed for a fixed σ_0 iff it satisfies, for all $(t,y) \in (0,T)\times\mathbb{R}^{d-1}$:

$$\sigma_0(t,y) \in \Sigma(t,y),$$

with the linear subspace Σ defined by:

$$\Sigma := \left((G_d^+|_{x=0})^{-1} - (G_d^-|_{x=0})^{-1}\right)\left(\mathbb{E}_-(G_d^+|_{x=0})\bigcap\mathbb{E}_+(G_d^-|_{x=0})\right).$$

The equation giving U_0^* has a unique solution iff:

$$\left[v \in \mathbb{E}_-\left(G_d^+(t,y,0)\right)\bigcap\mathbb{E}_+\left(G_d^-(t,y,0)\right), \left(G_d^+(t,y,0) - G_d^-(t,y,0)\right)v = 0\right] \Rightarrow [v=0],$$

which is equivalent to:

$$\dim\Sigma = \dim\ \mathbb{E}_-(G_d^+|_{x=0})\bigcap\mathbb{E}_+(G_d^-|_{x=0}).$$

This property results from our assumptions, as we will prove now. As we shall see below, due to the Uniform Evans Condition holding, one gets:

$$\dim\Sigma = N - \dim\mathbb{E}_-(A_d^-|_{x=0}) - \dim\mathbb{E}_+(A_d^+|_{x=0}).$$

Since $A_d^-|_{x=0}$ and $A_d^+|_{x=0}$ are nonsingular, $\dim\mathbb{E}_-(A_d^-|_{x=0}) = N - \dim\mathbb{E}_+(A_d^-|_{x=0})$ and $\dim\mathbb{E}_+(A_d^+|_{x=0}) = N - \dim\mathbb{E}_-(A_d^+|_{x=0})$. Plus, by Lemma 3.2.7, we have $\dim\mathbb{E}_-(G_d^+|_{x=0}) = \dim\mathbb{E}_-(A_d^+|_{x=0})$ and $\dim\mathbb{E}_+(G_d^-|_{x=0}) = \dim\mathbb{E}_+(A_d^-|_{x=0})$. We obtain thus:

$$N + \dim\Sigma = \dim\mathbb{E}_+(G_d^-|_{x=0}) + \dim\mathbb{E}_-(G_d^+|_{x=0}).$$

Thanks to our transversality assumption stated in Assumption 3.2.6, there holds:

$$\dim \mathbb{E}_+(G_d^-|_{x=0}) + \dim \mathbb{E}_-(G_d^+|_{x=0}) = N + \dim\ \mathbb{E}_-(G_d^+|_{x=0}) \bigcap \mathbb{E}_+(G_d^-|_{x=0}).$$

This ends the proof of:

$$\dim \Sigma = \dim\ \mathbb{E}_-(G_d^+|_{x=0}) \bigcap \mathbb{E}_+(G_d^-|_{x=0}).$$

We must however know $\sigma_0(t,y) \in \Sigma(t,y)$ in order to obtain U_0^*.
σ_0 is deduced from the computation of the profile $\underline{U}_0$, which is solution of the following mixed hyperbolic problem:

$$(3.2.4) \qquad \begin{cases} \widetilde{\mathcal{H}}\underline{U}_0 = \widetilde{f}, & \{x > 0\}, \\ \underline{U}_0^+|_{x=0} - \underline{U}_0^-|_{x=0} \in \Sigma, \\ \underline{U}_0|_{t<0} = 0 & . \end{cases}$$

We will now sketch a proof of the well-posedness of this equation. Some elements of it will be proved afterwards, in another subsection.
The function $\underline{U}_0$ is also solution of the mixed hyperbolic problem:

$$\begin{cases} \widetilde{\mathcal{H}}\underline{U}_0 = \widetilde{f}, & \{x > 0\}, \\ \Gamma^H \underline{U}_0|_{x=0} = 0, \\ \underline{U}_0|_{t<0} = 0, \end{cases}$$

where Γ^H denotes a linear operator such that:

$$\ker \Gamma^H = \mathcal{C}(t,y) := \left\{ \begin{pmatrix} U_0^{*+}|_{(z,x)=0} \\ U_0^{*-}|_{(z,x)=0} \end{pmatrix} \ : \ U_0^{*+}|_{(z,x)=0} - U_0^{*-}|_{(z,x)=0} \in \Sigma \right\};$$

note that $\mathcal{C}$ is the stable manifold for the dynamical system U_0^* is solution of. The Uniform Lopatinski Condition means that there is $C > 0$, such that, for all $(t,y) \in (0,T) \times \mathbb{R}^{d-1}$ and ζ with $\gamma > 0$, there holds:

$$det(\mathbb{E}_+(A|_{x=0^-}), \mathbb{E}_-(A|_{x=0^+}))) \geq C > 0,$$

where we recall that:

$$A(t,y,x;\zeta) := -(A_d)^{-1}(t,y,x) \left((i\tau + \gamma) A_0(t,y,x) + i \sum_{j=1}^{d-1} \eta_j A_j(t,y,x) \right).$$

In particular, taking $\gamma = 1$ and $(\tau, \eta) = 0$, it induces that:

$$\mathbb{E}_-(A_d^-|_{x=0}) \bigcap \mathbb{E}_+(A_d^+|_{x=0}) = \{0\}.$$

We will prove in section 3.2.5 that this Uniform Lopatinski Condition holds. It is a result very similar to the one of Rousset in [Rou03], established in the nonlinear framework, which states that the Uniform Lopatinski Condition holds for the limiting hyperbolic problem as the consequence of the Uniform Evans condition holding for the parabolic, viscously perturbed, problem. We underline that, in our case, our transversality assumption is necessary in order to prove this result. Remark that the Uniform Lopatinski Condition holds iff there is $C > 0$ such that, for all $(t, y) \in (0, T) \times \mathbb{R}^{d-1}$ and ζ with $\gamma > 0$, there holds:

$$det \left(\mathbb{E}_+(A|_{x=0^-}) \bigoplus \mathbb{E}_-(A|_{x=0^+})(t, y, \zeta), \Sigma(t, y) \right) \geq C > 0.$$

It implies that $\dim \Sigma = N - \dim \mathbb{E}_-(A|_{x=0^+}) - \dim \mathbb{E}_+(A|_{x=0^-})$. Due to our hyperbolicity assumption, $\dim \mathbb{E}_-(A|_{x=0^+}) = \dim \mathbb{E}_+(A_d^+|_{x=0})$ and $\dim \mathbb{E}_+(A|_{x=0^-}) = \dim \mathbb{E}_-(A_d^-|_{x=0})$. Hence $\dim \Sigma = N - \dim \mathbb{E}_-(A_d^-|_{x=0}) - \dim \mathbb{E}_+(A_d^+|_{x=0})$. Remark that, in the case of a 1-D problem with a piecewise constant coefficient, equal to $A^\pm$ on $\{\pm x > 0\}$, taking $B = Id$ as the viscosity tensor, the Uniform Lopatinski Condition writes:

$$\mathbb{E}_-(A^-) \bigoplus \mathbb{E}_+(A^+) \bigoplus \Sigma := \mathbb{R}^N.$$

For the sake of completeness, we will now show that the construction of the profiles can go on at any order. Let us assume the profiles up to order $n - 1$, with $n \leq M$, have been computed. We will now proceed with the construction of the profiles $\underline{U}_n$ and U_n^*.
To begin with, U_n^* satisfies the ODE in z :

$$\begin{cases} \widetilde{A}_d(t, y, x)\partial_z U_n^* - \widetilde{B}_{d,d}(t, y, x)\partial_z^2 U_n^* = \varphi_n^*, \\ U_n^{*+}|_{(z,x)=0} - U_n^{*-}|_{(z,x)=0} = -\sigma_n := -\left(\underline{U}_n^+|_{x=0} - \underline{U}_n^-|_{x=0}\right), \\ \partial_z U_n^{*+}|_{(z,x)=0} + \partial_z U_n^{*-}|_{(z,x)=0} = -\left(\partial_x \underline{U}_{n-1}^+|_{x=0} + \partial_x \underline{U}_{n-1}^-|_{x=0}\right), \end{cases}$$

with

$$\varphi_n^* = -\partial_t U_{n-1}^* - \sum_{j=1}^{d-1} \widetilde{A}_j \partial_j U_{n-1}^* + \sum_{j=1}^{d} \partial_j (B_{j,d} \partial_z U_{n-1}^*)$$

$$+\sum_{k=1}^{d}\partial_z(B_{d,k}\partial_k U_{n-1}^*)+\sum_{j,k<d}\partial_j(B_{j,k}\partial_k U_{n-2}^*).$$

As a consequence, there is $v_n^* \in e^{-\delta z}H^\infty((0,T)\times\mathbb{R}^{d-1}\times\mathbb{R}_+^*\times\mathbb{R}_+^*)$ such that:

$$U_n^*(t,y,x,z)=e^{\widetilde{G}_d(t,y,x)z}\,(U_n^*|_{z=0}-v_n^*|_{z=0})+v_n^*(t,y,x,z).$$

Some more computations show that the ODE giving U_n^{*-} is well-posed for fixed σ_n, provided that σ_n belongs to Σ_n, where Σ_n is an affine space directed by Σ. More precisely, Σ_n writes:

$$\Sigma_n = q_n + \Sigma,$$

with $q_n \in H^\infty((0,T)\times\mathbb{R}^{d-1})$.
$\underline{U}_n$ is then solution of the mixed hyperbolic problem satisfying a Uniform Lopatinski Condition:

$$\text{(3.2.5)}\qquad \begin{cases} \widetilde{\mathcal{H}}\underline{U}_n = \displaystyle\sum_{1\le j,k\le d}\partial_j(B_{j,k}\partial_k\underline{U}_{n-1}), & \{x>0\},\\ \underline{U}_n^+|_{x=0}-\underline{U}_n^-|_{x=0}\in\Sigma_n, & \\ \underline{U}_n|_{t<0}=0. & \end{cases}$$

Indeed, there is $r_n \in H^\infty((0,T)\times\mathbb{R}^{d-1})$, such that the problem writes as well:

$$\begin{cases} \widetilde{\mathcal{H}}\underline{U}_n = \displaystyle\sum_{1\le j,k\le d}\partial_j(B_{j,k}\partial_k\underline{U}_{n-1}), & \{x>0\},\\ \Gamma^H\underline{U}_n|_{x=0}=\Gamma^H r_n, & \\ \underline{U}_n|_{t<0}=0. & \end{cases}$$

$\sigma_n \in \Sigma_n$ is deduced from this equation and thus U_n^* can now be computed.

3.2.3 Stability Analysis and Main Result.

The error equation writes, for $w^\varepsilon = u^\varepsilon_{app}-u^\varepsilon$:

$$\text{(3.2.6)}\qquad \begin{cases} \mathcal{H}^\varepsilon w^\varepsilon = \varepsilon^M R^\varepsilon,\\ w^\varepsilon|_{t<0}=0. \end{cases}$$

Our goal here is to prove that the error w^ε converges towards zero as the viscosity vanishes. To be more precise we will prove some uniform energy estimates in L^2 norm. The proof of these stability estimates is almost the same as the ones performed in [MZ05]. In [Mét04], Métivier gives a simplified version of the proof for constant coefficients. Assuming the coefficients are constant, the energy estimates can then be proved by performing a tangential Laplace-Fourier transform of the problem. In this special case, the symmetrizers are Fourier Multipliers hence avoiding the need of any pseudodifferential calculus. Moreover, we emphasize that the analysis of the stability of the problem for frozen coefficients is a crucial step in the proof of more general energy estimates.

For our part, some elements of proof have to be given since our assumptions differ of the ones in [Mét04] or in [MZ05]. In order to shorten a not so original proof, we will rather focus on showing that the scheme of proof exposed in [MZ05] works for our present problem. We will proceed to do so on a very simplified example. In the process, we will reinvestigate the link existing between the Uniform Evans Condition holding and the construction of Kreiss-type symmetrizers.
Our proof will be performed in the 1-D framework, for piecewise constant coefficients and for a viscosity tensor $B = Id$. Rather than giving a proof more simple but also more specific to our example, we aim at giving an easily generalized proof, which, even if exposed differently, relates clearly to [Mét04], [MZ05] and [GMWZ05].
Note that a similar proof of stability can be proved in the multi-D framework thanks to the Theorem 3.2.12, which states the existence of a low frequency symmetrizer ([Mét04]), be it for 1-D or multi-D systems. Remark that, in our special case, no glancing modes (i.e eigenvalues which becomes, after a rescaling focused on a neighborhood of $\zeta = 0$, purely imaginary and not semi-simple) appear, which makes the proof of Theorem 3.2.12 become a lot easier to perform.
These stability results can also be proved for multi-D systems with piecewise smooth coefficients, constant outside a compact set, through the use of pseudodifferential calculus.
Let us now state the results obtained under our initial assumptions: choosing M big enough, we get:

Theorem 3.2.8 (Stability)**.** *There is* $C > 0$ *such that, for all* $0 < \varepsilon <$

1, *there holds*

$$\|u^\varepsilon - u^\varepsilon_{app}\|_{L^2((0,T)\times\mathbb{R}^d)} \leq C\varepsilon.$$

Let u be $u := u^+\mathbf{1}_{x\geq 0}+u^-\mathbf{1}_{x<0}$, where (u^+, u^-) is the unique solution of the well-posed transmission problem:

$$\begin{cases} \mathcal{H}^+u^+ = f^+, & \{x>0\}, \\ \mathcal{H}^-u^- = f^-, & \{x>0\}, \\ u^+|_{x=0} - u^-|_{x=0} \in \Sigma, \\ u^+|_{t<0} = 0, \quad u^-|_{t<0} = 0. \end{cases} \tag{3.2.7}$$

We obtain then the following convergence result, which is our main result:

Theorem 3.2.9 (Convergence)**.** *There is $C > 0$ such that, for all $0 < \varepsilon < 1$, there holds:*

$$\|u^\varepsilon - u\|_{L^2((0,T)\times\mathbb{R}^d)} \leq C\varepsilon.$$

3.2.4 Simplified proof of stability estimates.

We will prove stability estimates for the following viscous system in one space dimension:

$$\begin{cases} \partial_t u^\varepsilon + A(x)\partial_x u^\varepsilon - \varepsilon\partial_x^2 u^\varepsilon = f, & (t,x) \in (0,T)\times\Omega, \\ u^\varepsilon|_{t<0} = 0. \end{cases}$$

where the coefficient A is assumed piecewise constant, equal to A^+ on $\{x > 0\}$ and equal to A^- on $\{x < 0\}$. We still make the same assumptions as before on this system. We have constructed

$$u^\varepsilon_{app} := u^{\varepsilon+}_{app}(t,x)\mathbf{1}_{x>0} + u^{\varepsilon-}_{app}(t,-x)\mathbf{1}_{x<0}$$

such that, if we denote $w^\varepsilon = u^\varepsilon_{app} - u^\varepsilon$, there holds:

$$\begin{cases} \partial_t w^\varepsilon + A(x)\partial_x w^\varepsilon - \varepsilon\partial_x^2 w^\varepsilon = \varepsilon^M R^\varepsilon, & (t,x) \in \Omega, \\ w^\varepsilon|_{t<0} = 0. \end{cases}$$

where $\Omega = (0,T)\times\mathbb{R}$, R^ε belongs to $H^\infty((0,T)\times\mathbb{R}^*)$ and vanishes in the past. Since our method of estimation comes from pseudodifferential calculus, we have to perform a tangential Fourier-Laplace transform of the problem. To this aim, it is necessary to extend the definition of our error, in order for it to be defined for all time $t \in \mathbb{R}$. We denote by $\underline{\widetilde{R}}^\varepsilon$, R^ε extended by 0 outside $(-\infty, T) \times \mathbb{R}$. Let us now proceed with the extension of our error to $t \geq T$. We call by $\underline{\widetilde{w}}^\varepsilon$ the unique solution of:

$$\begin{cases} \mathcal{H}\underline{\widetilde{w}}^\varepsilon - \varepsilon\partial_x^2\underline{\widetilde{w}}^\varepsilon = \varepsilon^M \underline{\widetilde{R}}^\varepsilon, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ \underline{\widetilde{w}}^\varepsilon|_{t<0} = 0. \end{cases} \tag{3.2.8}$$

Note well that the restriction of $\underline{\widetilde{w}}^\varepsilon$ to Ω is w^ε. For the sake of simplicity, we will still denote $\underline{\widetilde{w}}^\varepsilon$ [resp $\underline{\widetilde{R}}^\varepsilon$] by w^ε [resp R^ε] in what follows.
We now come back to our error equation (3.2.8). To begin with, let us rewrite the problem (3.2.8) in a convenient form. w^ε is solution of:

$$\partial_t w^\varepsilon + A(x)\partial_x w^\varepsilon - \varepsilon\partial_x^2 w^\varepsilon = \varepsilon^M R^\varepsilon, \qquad (t,x) \in \mathbb{R} \times \mathbb{R},$$

Let γ stand for a positive parameter. We denote then by $\hat{w}^{\varepsilon\pm} := \mathcal{F}(e^{-\gamma t} w^{\varepsilon\pm})$ and $\hat{R}^{\varepsilon\pm} := \mathcal{F}(e^{-\gamma t} R^{\varepsilon\pm})$, where $\mathcal{F}$ stands for the tangential Fourier transform (with respect to t) and the $\pm$ superscripts indicates restrictions to $\{\pm x > 0\}$, we have then:

$$\begin{cases} (i\tau + \gamma)\hat{w}^{\varepsilon+} + A^+\partial_x\hat{w}^{\varepsilon+} - \varepsilon\partial_x^2\hat{w}^{\varepsilon+} = \varepsilon^M \hat{R}^{\varepsilon+}, \quad \{x > 0\}, \\ (i\tau + \gamma)\hat{w}^{\varepsilon-} + A^-\partial_x\hat{w}^{\varepsilon-} - \varepsilon\partial_x^2\hat{w}^{\varepsilon-} = \varepsilon^M \hat{R}^{\varepsilon-}, \quad \{x < 0\}, \\ \hat{w}^{\varepsilon+}|_{x=0} - \hat{w}^{\varepsilon-}|_{x=0} = 0, \\ \partial_x\hat{w}^{\varepsilon+}|_{x=0} - \partial_x\hat{w}^{\varepsilon-}|_{x=0} = 0. \end{cases} \tag{3.2.9}$$

Remark that, by taking γ big enough, the restrictions of the solution w^ε of (3.2.8) to $\{\pm x > 0\}$ are given by:

$$w^{\varepsilon\pm} = e^{\gamma t}\mathcal{F}^{-1}(\hat{w}^{\varepsilon\pm}),$$

where $(\hat{w}^{\varepsilon+}, \hat{w}^{\varepsilon-})$ are the solutions of the transmission problem (3.2.9).

Taking $W^{\varepsilon\pm}(i\tau + \gamma, x) = \begin{pmatrix} \hat{w}^{\varepsilon\pm} \\ \\ \varepsilon\partial_x\hat{w}^{\varepsilon\pm} \end{pmatrix}$, we have then:

$$\begin{cases} \partial_x W^{\varepsilon+} = \begin{pmatrix} \partial_x \hat{w}^{\varepsilon+} \\ \\ \varepsilon \partial_x^2 \hat{w}^{\varepsilon+} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon} Id \\ \\ (i\tau+\gamma) & \frac{1}{\varepsilon} A^+ \end{pmatrix} \begin{pmatrix} \hat{w}^{\varepsilon+} \\ \\ \varepsilon \partial_x \hat{w}^{\varepsilon+} \end{pmatrix} + \begin{pmatrix} 0 \\ \\ \varepsilon^M \hat{R}^{\varepsilon+} \end{pmatrix}, \\ \\ \partial_x W^{\varepsilon-} = \begin{pmatrix} \partial_x \hat{w}^{\varepsilon-} \\ \\ \varepsilon \partial_x^2 \hat{w}^{\varepsilon-} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon} Id \\ \\ (i\tau+\gamma) & \frac{1}{\varepsilon} A^- \end{pmatrix} \begin{pmatrix} \hat{w}^{\varepsilon-} \\ \\ \varepsilon \partial_x \hat{w}^{\varepsilon-} \end{pmatrix} + \begin{pmatrix} 0 \\ \\ \varepsilon^M \hat{R}^{\varepsilon-} \end{pmatrix}, \\ W^{\varepsilon+}|_{x=0} - W^{\varepsilon-}|_{x=0} = 0. \end{cases}$$

We note $\zeta = (\tau, \gamma)$ and $\widetilde{\zeta} = (\varepsilon\tau, \varepsilon\gamma)$. Multiplying the previous equation by ε gives:

$$\begin{cases} \partial_z W^{\varepsilon+} - \mathbb{A}^+(\widetilde{\zeta}) W^{\varepsilon+} = G^+, \quad \{z > 0\}, \\ \partial_z W^{\varepsilon-} - \mathbb{A}^-(\widetilde{\zeta}) W^{\varepsilon-} = \widetilde{G}^-, \quad \{z < 0\}, \\ W^{\varepsilon+}|_{z=0} = W^{\varepsilon-}|_{z=0}, \end{cases}$$

where $\mathbb{A}^{\pm}(\widetilde{\zeta}) = \begin{pmatrix} 0 & Id \\ (i\widetilde{\tau} + \widetilde{\gamma}) Id & A^{\pm} \end{pmatrix}$ and $G^{\pm} = \begin{pmatrix} 0 \\ \varepsilon^{M+1} \hat{R}^{\varepsilon\pm} \end{pmatrix}$, and z stands for the fast variable $\frac{x}{\varepsilon}$. Note that the first energy estimates to be proved will concern this equation.

Proof of the error estimate by symmetrizers

We will now show how, thanks to the Uniform Evans condition holding, stability estimates can be proved by symmetrizers for the three different regimes of frequency: low, medium and high. In the construction of symmetrizers, for the sake of simplicity, we will drop the tildes in our notations and only introduce them back when needed.

An error estimate for medium frequencies

For $1 \leq |\zeta| \leq 2$, we will prove here Proposition 3.2.10. Denote $\widetilde{\mathbb{A}}^- = -\mathbb{A}^-$, $\underline{W}^{\varepsilon-} := W^{\varepsilon-}(t, -z)$ and $\underline{G}^- = \widetilde{G}^-(t, -z)$, $\underline{W}^{\varepsilon-}$ satisfies then the following ODE in z:

$$\begin{cases} \partial_z \underline{W}^{\varepsilon-} - \widetilde{\mathbb{A}}^- \underline{W}^{\varepsilon-} = \underline{G}^-, \quad \{z > 0\}, \\ \lim\limits_{z \to \infty} \underline{W}^{\varepsilon-} = 0. \end{cases}$$

It implies that $\underline{W}^{\varepsilon-}|_{z=0}$ belongs to the stable manifold:

$$\mathcal{W}^{s-} = q_n^-|_{z=0} + \mathbb{E}_- \left(\widetilde{\mathbb{A}}^-|_{z=0} \right),$$

where q_n^- is a bounded solution of the above ODE. Even if q_n^- can be chosen in several ways, the space $\mathcal{W}^{s-}$ is uniquely defined.
In addition, $W^{\varepsilon+}$ is solution of:

$$\begin{cases} \partial_z W^{\varepsilon+} - \mathbb{A}^+ W^{\varepsilon+} = G^+, & \{z > 0\}, \\ \lim\limits_{z\to\infty} W^{\varepsilon+} = 0. \end{cases}$$

Therefore $W^{\varepsilon+}|_{z=0}$ belongs to the stable manifold:

$$\mathcal{W}^{s+} = q_n^+|_{z=0} + \mathbb{E}_- \left(\mathbb{A}^+|_{z=0} \right).$$

We have

$$\mathbb{C}^{2N} = \mathbb{E}_-(\mathbb{A}^+) \bigoplus \mathbb{E}_+(\mathbb{A}^+).$$

The projectors associated to this decomposition will respectively be Π_1^- and Π_1^+. Under our structure assumptions, as in [Mét04], there is two hermitian symmetric, uniformly bounded, matrices S_1^+ and S_1^- such that:

- There is $C > 0$ such that, for all $q \in \mathbb{E}_+(\mathbb{A}^+)$,

  $$\langle \Re e S_1^+ \mathbb{A}^+ q, q \rangle \geq C|q|^2,$$

  and, for all $q \in \mathbb{E}_-(\mathbb{A}^+)$,

  $$-\langle \Re e S_1^- \mathbb{A}^+ q, q \rangle \geq C|q|^2.$$

- There is $c_1^+ > 0$ and $c_1^- > 0$ such that:

  $$\Pi_1^{+*}\Pi_1^+ \leq S_1^+ \leq c_1^+ \Pi_1^{+*}\Pi_1^+, \quad \Pi_1^{-*}\Pi_1^- \leq S_1^- \leq c_1^- \Pi_1^{-*}\Pi_1^-.$$

Note well that neither the Uniform Evans condition, nor our boundary conditions intervene in the proof of this result. In what follows, κ will always denote a positive parameter. We define then $\mathcal{S}_\kappa^+$ by

$$\mathcal{S}_\kappa^+ := \kappa S_1^+ - S_1^-.$$

We will prove further that, provided that we choose κ large enough, $\mathcal{S}_\kappa^+$ is a suitable Kreiss-type symmetrizer for our system if the Uniform Evans Condition hold. For now, we have constructed a hermitian symmetric, uniformly bounded matrix $\mathcal{S}_\kappa^+$ and there is $c_{1,\kappa} > 0$ such that:

$$2\Re e \mathcal{S}_\kappa^+ \mathbb{A}^+ \geq c_{1,\kappa} Id.$$

As we will see, our stability condition will play a role in the control of the traces $W^{\varepsilon+}|_{z=0}$ and $\underline{W}^{\varepsilon-}|_{z=0}$, which is the crucial step in the proof of our energy estimates. Those traces are linked together by the relations: $W^{\varepsilon+}|_{z=0} = \underline{W}^{\varepsilon-}|_{z=0}$, with $W^{\varepsilon+}|_{z=0} \in \mathcal{W}^{s+}$ and $\underline{W}^{\varepsilon-}|_{z=0} \in \mathcal{W}^{s-}$. Remark that there is uniqueness for the traces $W^{\varepsilon+}|_{z=0} = \underline{W}^{\varepsilon-}|_{z=0}$, satisfying the above relations, iff:

$$\mathbb{E}_-\left(\widetilde{\mathbb{A}}^+|_{z=0}\right) \bigcap \mathbb{E}_-\left(\widetilde{\mathbb{A}}^-|_{z=0}\right) = \{0\},$$

which is equivalent, for the range of frequencies we are presently considering, to our Uniform Evans Condition.

We perform an analogous construction of a potential symmetrizer for $\underline{W}^{\varepsilon-}$. The projectors associated to the decomposition:

$$\mathbb{C}^{2N} = \mathbb{E}_-(\widetilde{\mathbb{A}}^-) \bigoplus \mathbb{E}_+(\widetilde{\mathbb{A}}^-)$$

will respectively be Π_2^- and Π_2^+.

Under our structure assumptions, as in [Mét04], there is two hermitian symmetric, uniformly bounded, matrices S_2^+ and S_2^- such that:

- There is $C > 0$ such that, for all $q \in \mathbb{E}_+(\widetilde{\mathbb{A}}^-)$,

  $$\langle \Re e S_2^+ \widetilde{\mathbb{A}}^- q, q\rangle \geq C|q|^2,$$

  and, for all $q \in \mathbb{E}_-(\widetilde{\mathbb{A}}^-)$,

  $$-\langle \Re e S_2^- \widetilde{\mathbb{A}}^- q, q\rangle \geq C|q|^2.$$

- There is $c_2^+ > 0$ and $c_2^- > 0$ such that:

  $$\Pi_2^{+*}\Pi_2^+ \leq S_2^+ \leq c_2^+ \Pi_2^{+*}\Pi_2^+, \quad \Pi_2^{-*}\Pi_2^- \leq S_2^- \leq c_2^- \Pi_2^{-*}\Pi_2^-.$$

Like before, neither our stability condition, nor our boundary conditions intervene here. We define then $\mathcal{S}^-_\kappa$ by

$$\mathcal{S}^-_\kappa := \kappa S_2^+ - S_2^-.$$

The so constructed matrix $\mathcal{S}^-_\kappa$ is hermitian symmetric, uniformly bounded and satisfies, for some $c_{2,\kappa} > 0$:

$$2\Re e \mathcal{S}^-_\kappa \mathbb{A}^- \geq c_{2,\kappa} Id.$$

We recall that $W^{\varepsilon +}|_{z=0} = \underline{W}^{\varepsilon -}|_{z=0} = \underline{q}$. For the sake of clarity, we will drop the κ subscripts. Let us now prove our energy estimates.

$$-\left\langle \mathcal{S}^+ W^{\varepsilon +}|_{z=0}, W^{\varepsilon +}|_{z=0}\right\rangle = \int_0^\infty \left\langle \mathcal{S}^+ \frac{d}{dz} W^{\varepsilon +}, W^{\varepsilon +}\right\rangle + \left\langle \mathcal{S}^+ W^{\varepsilon +}, \frac{d}{dz} W^{\varepsilon +}\right\rangle \, dz$$

$$= \int_0^\infty \left\langle 2\Re e \mathcal{S}^+ \mathbb{A}^+ W^{\varepsilon +}, W^{\varepsilon +}\right\rangle \, dz + \int_0^\infty \left\langle 2\Re e \mathcal{S}^+ \widetilde{G}^+, W^{\varepsilon +}\right\rangle \, dz$$

thus

$$c_1 \int_0^\infty \left\langle W^{\varepsilon +}, W^{\varepsilon +}\right\rangle \, dz \leq -\left\langle \mathcal{S}^+ W^{\varepsilon +}|_{z=0}, W^{\varepsilon +}|_{z=0}\right\rangle + \left| \int_0^\infty \left\langle 2\Re e \mathcal{S}^+ \widetilde{G}^+, W^{\varepsilon +}\right\rangle \, dz \right|$$

Denoting by $\|u\| := \|u\|_{L^2(\mathbb{R}_+^*)} = \left(\int_0^\infty \langle u, u\rangle \, dz\right)^{\frac{1}{2}}$, we obtain then that there are $c_1' > 0$ and $C_1' > 0$ such that:

$$c_1' \|W^{\varepsilon +}\|^2 \leq -\left\langle \mathcal{S}^+ W^{\varepsilon +}|_{z=0}, W^{\varepsilon +}|_{z=0}\right\rangle + C_1' \|\Re e \mathcal{S}^+ \widetilde{G}^+\|^2.$$

Performing the same steps once again, we get that:

$$c_2' \|\underline{W}^{\varepsilon -}\|^2 \leq -\left\langle \mathcal{S}^- \underline{W}^{\varepsilon -}|_{z=0}, \underline{W}^{\varepsilon -}|_{z=0}\right\rangle + C_2' \|\Re e \mathcal{S}^- \underline{G}^-\|^2.$$

Taking $c = min(c_1', c_2')$, and $C = min(C_1', C_2')$, we get then:

$$c\|W^\varepsilon\|^2_{L^2(\mathbb{R})} + \left\langle (\mathcal{S}^+ + \mathcal{S}^-)\underline{q}, \underline{q}\right\rangle \leq C \left(\|\Re e \mathcal{S}^+ \widetilde{G}^+\|^2 + \|\Re e \mathcal{S}^- \widetilde{G}^-\|^2 \right).$$

Proposition 3.2.10. *For κ large enough, there is $\delta > 0$ such that, for all $\underline{q} \in \mathbb{C}^{2N}$, there holds:*

$$\left\langle \left(\mathcal{S}^+_\kappa + \mathcal{S}^-_\kappa\right) \underline{q}, \underline{q}\right\rangle \geq \delta \langle \underline{q}, \underline{q}\rangle. \tag{3.2.10}$$

Moreover, there is c, δ and C positive such that, for all $0 < \varepsilon < 1$, we have:

$$c\|W^\varepsilon\|^2_{L^2(\mathbb{R})} + \delta |W^\varepsilon|_{z=0}|^2 \leq C\|G\|^2_{L^2(\mathbb{R})}. \tag{3.2.11}$$

Proof. As a preliminary, we have the next lemma:

Lemma 3.2.11. *Suppose the uniform Evans condition satisfied, then, for all* $|\zeta| \neq 0$ *and for all* $\underline{q} \in \mathbb{C}^{2N}$*, we have either* $\underline{q} = 0$ *or* $\Pi_1^+(\zeta)\underline{q} \neq 0$ *or* $\Pi_2^+(\zeta)\underline{q} \neq 0$.

Proof. Indeed, fixing $\zeta \neq 0$, if there exists $\underline{q} \neq 0$ such that $\Pi_1^+\underline{q} = 0$ or $\Pi_2^+\underline{q} = 0$, we get:

$$\Pi_1^-(\underline{q}) = \Pi_2^-(\underline{q}) = \underline{q}.$$

As a result $\underline{q}$ is nonzero and belongs to $\mathbb{E}_-(\mathbb{A}^+) \bigcap \mathbb{E}_+(\mathbb{A}^+)$, which contradicts our stability assumption. □

For $\underline{q} = 0$, the inequality is trivially satisfied. For $\underline{q} \in \mathbb{C}^{2N}$ such that $\Pi_1^+\underline{q} \neq 0$, taking κ large enough gives the result. Notice that, for $\underline{q} \in \mathbb{C}^{2N}$ with $\Pi_2^+\underline{q} \neq 0$, taking κ large enough also leads to the result. Now Lemma 3.2.11 states that either $q = 0$, either $\Pi_1^+\underline{q} \neq 0$ or $\Pi_2^+\underline{q} \neq 0$, which achieves the proof of the first part of Proposition 3.2.10, using the inequality (3.2.10), it follows that:

$$c\|W^\varepsilon\|^2_{L^2(\mathbb{R})} + \delta|W^\varepsilon|_{z=0}|^2 \leq C\left(\|\Re e\mathcal{S}^+\widetilde{G}^+\|^2_{L^2(\mathbb{R}^+)} + \|\Re e\mathcal{S}^-\underline{G}^-\|^2_{L^2(\mathbb{R}^-)}\right),$$

thus leading to the estimate (3.2.11).

□

An error estimate for high frequencies

Denote by

$$w_1^{\varepsilon+} := \begin{pmatrix} \Lambda\hat{w}^{\varepsilon+} \\ \partial_z\hat{w}^{\varepsilon+} \end{pmatrix},$$

and

$$w_1^{\varepsilon-} := \begin{pmatrix} \Lambda\hat{w}^{\varepsilon-} \\ \partial_z\hat{w}^{\varepsilon-} \end{pmatrix},$$

then, for Λ big enough, our problem is transformed in the study, for $\zeta \in \{|\zeta| = 1\} \bigcup \{\gamma \geq 0\}$ of the same equations than for medium frequencies, this time with unknown $(w_1^{\varepsilon+}, w_1^{\varepsilon-})$ instead of $(W^{\varepsilon+}, W^{\varepsilon-})$. We note $w_1^\varepsilon = w_1^{\varepsilon+}\mathbf{1}_{x>0} + w_1^{\varepsilon-}\mathbf{1}_{x<0}$. We obtain, the same way as for medium frequencies, that there are $c_h > 0$ and $\delta_h > 0$ such that for for all $|\zeta| > 2$ and for all $0 < \varepsilon < 1$, there holds:

(3.2.12)
$$c_h \left\|w_1^\varepsilon\right\|_{L^2(\mathbb{R})}^2+\delta_h \left|w_1^\varepsilon|_{x=0}\right|^2 \leq C\left(\|Re\mathcal{S}^+\widetilde{G}^+\|^2+\|Re\mathcal{S}^-\underline{G}^-\|^2\right).$$

An error estimate for low frequencies

For low frequencies, the study becomes much more delicate since some eigenvalues of $\mathbb{A}^\pm$ does not stay away from the imaginary axis, asymptotically when ζ tends to zero. As a result, the spectral projectors on the negative or positive eigenspaces of $\mathbb{A}^+$ and $\mathbb{A}^-$, which are needed in the construction of the symmetrizers are no longer well-defined. Hence, an appropriate rescaling has to be introduced for ζ in a neighborhood of zero, the important linear subspaces to consider are then the positive and negative spaces of the rescaled versions of $\mathbb{A}^+$ and $\mathbb{A}^-$. After rescaling, the spectral projectors on these spaces become perfectly well-defined, for $\check{\tau}^2+\check{\gamma}^2=1$ and $\check{\gamma}>0$, where $\check{\tau}=\frac{\tau}{|\zeta|}$ and $\check{\gamma}=\frac{\tau}{|\gamma|}$ are the frequencies rescaled for a low frequency analysis. A logical idea would be to prove a continuous extension of these linear subspaces to $\{\check{\gamma}=0\}$, in order to help with the construction of a low frequency symmetrizers. However, what happens is the converse, since the fact that those linear subspaces extends continuously to $\{\check{\gamma}=0\}$ is a consequence of the construction of a Kreiss-type symmetrizer for low frequencies as defined by Theorem 3.2.12. This is shown in [MZ04].
Let us now give a brief overview of the low frequency analysis of the problem. By a suitable change of basis, the matrix $\mathbb{A}^\pm$ becomes block diagonal. Constructing a symmetrizer for $\mathbb{A}^\pm$ reduces to the construction of a symmetrizer for each diagonal blocks. We group together the eigenvalues which do not come near the imaginary axis, forming what we will call the parabolic block. For this block, our treatment does not differ from the one previously described for medium frequencies. The other eigenvalues can be grouped together in the hyperbolic block. As explained in the beginning of this section, the construction of the symmetrizers for this hyperbolic block needs a specific approach. For 1-D systems, which is our present case, the construction of a low frequency symmetrizer is rather easy since all the eigenvalues in the hyperbolic block are strictly hyperbolic, which means that, even if they do cross the imaginary axis, they remain semi-simple. In general, for multi-D systems, glancing modes, that is to say purely imaginary, non semi-simple

eigenvalues also do appear. Those need an elaborate analysis. For those part of the analysis, we can rely on Theorem 3.2.12 proved for instance in [Mét04]. Indeed, compared to the problems studied in [Mét04], we make the same structure assumptions (hyperbolicity, parabolicity and hyperbolicity-parabolicity), even though, our boundary conditions, and therefore the expression of our Uniform Evans Condition differs. As a consequence, the results of [Mét04], proved by using only the structure assumptions, also holds here. It is in particular the case of Theorem 3.2.12.

$W^{\varepsilon+}$ and $\underline{W}^{\varepsilon-}$ satisfying almost the same equations, we will mostly describe the proof of the energy estimates involving $W^{\varepsilon+}$. Let us introduce some notations and some important properties involved in the low frequency study of the hyperbolic part. Using polar coordinates, we define:

$$\rho := |\tau + i\gamma|.$$

There is a nonsingular $N \times N$ matrix ν^+ and two $N \times N$ matrices H^+ and P^+, such that:

$$(\nu^+)^{-1}\mathbb{A}^+\nu^+ = \mathbb{A}_2^+ := \begin{pmatrix} H^+(\zeta) & 0 \\ 0 & P^+(\zeta) \end{pmatrix},$$

with the eigenvalues of P^+ staying away from he imaginary axis and the eigenvalues of H^+ vanishing for $|\zeta| = 0$. Indeed, $\mathbb{A}^\pm$ has got exactly N hyperbolic eigenvalues and N parabolic eigenvalues as proved for instance in [For07a]. In order to symmetrize properly H^+, we introduce the polar rescaling:

$$\zeta = \rho\check{\zeta} = \rho(\check{\tau}, \check{\gamma}),$$

we have thus $|\check{\zeta}| = 1$. The rescaled version of H^+, $\check{H}^+$ is then given by:

$$H^+(\zeta) = \rho\check{H}^+(\check{\zeta}, \rho).$$

Hence, $W_2^{\varepsilon+} = (\nu^+)^{-1}W^{\varepsilon+}$ satisfies the equation:

$$\begin{cases} \partial_z W_2^{\varepsilon+} - \mathbb{A}_2^+ W_2^{\varepsilon+} = (\nu^+)^{-1}\widetilde{G}^+, \quad \{z > 0\}, \\ W_2^{\varepsilon+}|_{z=0} = (\nu^+)^{-1}\underline{q} := \underline{q}_2 \quad . \end{cases}$$

The symmetrizer for this problem will then be constructed by block, as follows:

$$\mathcal{S}_l^+ = \begin{pmatrix} \rho\check{\mathcal{S}}_H^+(\check{\zeta}, \rho) & 0 \\ 0 & \mathcal{S}_P^+(\zeta) \end{pmatrix}.$$

The symmetrizer of P^+, $\mathcal{S}_P^+$ will not be detailed here since it is the exact analogous of the symmetrizer for medium frequencies.
For the hyperbolic part, we have:

$$\check{H}^+(\check{\zeta},0) = -(i\check{\tau}+\check{\gamma})(A^+)^{-1}.$$

For $\rho \geq C > 0$, H^+ has exactly N_1^+ eigenvalues with positive real part and N_1^- eigenvalues with negative real part while P^+ has exactly N_1^- eigenvalues with positive real part and N_1^+ eigenvalues with negative real part. For $\rho \geq C > 0$, we can construct $\mathcal{S}_H^+(\zeta) := \rho\check{\mathcal{S}}_H^+(\check{\zeta},\rho)$ the same way (we have the same qualitative behavior as for the medium frequencies previously treated). Under our assumptions, the following result, asserting that we can construct $\check{\mathcal{S}}_H^+(\check{\zeta},\rho)$, for $(\check{\zeta},\rho)$ in a neighborhood of $(\check{\zeta}_0,0)$ has been proved in [Mét04]:

Theorem 3.2.12. *For all* $\{|\check{\zeta}| = 1\} \bigcup \{\check{\gamma} \geq 0\}$, *there are two linear subspaces* $\underline{\mathbb{F}}_1^+$ *and* $\underline{\mathbb{F}}_1^-$ *of constant dimension satisfying:*

$$\mathbb{C}^N = \underline{\mathbb{F}}_1^+ \bigoplus \underline{\mathbb{F}}_1^-, \tag{3.2.13}$$

with $dim(\underline{\mathbb{F}}_1^+) = N_1^+$, $dim(\underline{\mathbb{F}}_1^-) = N_1^-$, *and such that for all* $\kappa_1 \geq 1$ *there exists a neighborhood* $\check{\omega}$ *of* $(\check{\zeta},0)$ *in* $\mathbb{R}^2 \times \mathbb{R}$, *a* C^∞ *mapping* $\check{\mathcal{S}}_H^+$ *from* $\check{\omega}$ *to the space of* $N \times N$ *matrices, and a constant* $c > 0$ *such that for all* $(\check{\zeta},\rho) \in \check{\omega}$,

$$\check{\mathcal{S}}_H^+(\check{\zeta},\rho) = \left(\check{\mathcal{S}}_H^+(\check{\zeta},\rho)\right)^*$$

for all $h \in \mathbb{C}^N$, *denoting by* $\underline{\Pi}_1^+$ *and* $\underline{\Pi}_1^-$ *the projectors associated to the decomposition* (3.2.13) *of* $\mathbb{C}^N$:

$$\left\langle \check{\mathcal{S}}_H^+(\check{\zeta},\rho)h,h \right\rangle \geq \kappa_1|\underline{\Pi}_1^+ h|^2 - |\underline{\Pi}_1^- h|^2$$

and, for all $(\check{\zeta},\rho) \in \check{\omega}$, *with* $\rho \geq 0$ *and* $\check{\gamma} \geq 0$:

$$2\Re e \left\langle \check{\mathcal{S}}_H^+(\check{\zeta},\rho)\check{H}^+(\check{\zeta},\rho)h,h \right\rangle \geq c(\check{\gamma}+\rho)|h|^2$$

Note that we have the analogous Theorem for $\underline{W}^{\varepsilon-}$:

Theorem 3.2.13. *For all* $\{|\check{\zeta}| = 1\} \bigcup \{\check{\gamma} \geq 0\}$, *there are two linear subspaces* $\underline{\mathbb{F}}_2^+$ *and* $\underline{\mathbb{F}}_2^-$ *of constant dimension satisfying:*

$$\mathbb{C}^N = \underline{\mathbb{F}}_2^+ \bigoplus \underline{\mathbb{F}}_2^-, \tag{3.2.14}$$

with $dim(\underline{\mathbb{F}}_2^+) = N_2^+$, $dim(\underline{\mathbb{F}}_2^-) = N_2^-$, and such that for all $\kappa_2 \geq 1$ there exists a neighborhood $\check{\omega}$ of $(\check{\zeta}, 0)$ in $\mathbb{R}^2 \times \mathbb{R}$, a C^∞ mapping $\check{\mathcal{S}}_H^-$ from $\check{\omega}$ to the space of $N \times N$ matrices, and a constant $c > 0$ such that for all $(\check{\zeta}, \rho) \in \check{\omega}$,

$$\check{\mathcal{S}}_H^-(\check{\zeta}, \rho) = \left(\check{\mathcal{S}}_H^-(\check{\zeta}, \rho)\right)^*$$

for all $h \in \mathbb{C}^N$, denoting by $\underline{\Pi}_2^+$ and $\underline{\Pi}_2^-$ the projectors associated to the decomposition (3.2.14) *of $\mathbb{C}^N$:*

$$\left\langle \check{\mathcal{S}}_H^-(\check{\zeta}, \rho)h, h \right\rangle \geq \kappa_2 |\underline{\Pi}_2^+ h|^2 - |\underline{\Pi}_2^- h|^2$$

and, for all $(\check{\zeta}, \rho) \in \check{\omega}$, with $\rho \geq 0$ and $\check{\gamma} \geq 0$:

$$2\Re e \left\langle \check{\mathcal{S}}_H^-(\check{\zeta}, \rho)\check{H}^-(\check{\zeta}, \rho)h, h \right\rangle \geq c(\check{\gamma} + \rho)|h|^2$$

We just expose here as a remark an important property linked to our current analysis.

Remark 3.2.14. *Let $\mathcal{H}^+(\zeta, \rho)$ be given by:*

$$\mathcal{H}^+(\zeta, \rho) = \check{H}^+(\check{\zeta}, \rho).$$

There exists $\mathrm{e}^+(\tau, \gamma, \xi, \rho)$ polynomial in ξ with smooth coefficients in (τ, γ, ρ) such that:

$$det\left((i\tau + \gamma)Id + i\xi A^+ + \rho Id\right) = \mathrm{e}^+(\tau, \gamma, \xi, \rho) det\left(i\xi Id - \mathcal{H}^+(\tau, \gamma, \rho)\right)$$

and $\mathrm{e}^+(\tau, \gamma, \xi, 0) \neq 0$. This shows the important link, for $\rho = 0$, existing between the spectral study of $\mathcal{H}^+$ and the spectral study of the symbol of the hyperbolic part of our equation.

For $\rho \geq 0$, we have, for all $h \in \mathbb{C}^N$:

$$2\Re e\langle \mathcal{S}_P^+ P^+ h, h\rangle \geq c\rho(\check{\gamma} + \rho)|h|^2.$$

As a result, For $\rho \geq 0$, we can construct $\mathcal{S}_l$ satisfying:

$$2\Re e\langle \mathcal{S}_l^+ \mathbb{A}^+ h, h\rangle \geq c(\gamma + \rho^2)|h|^2.$$

Mimicking what has been done for medium frequencies, after choosing for all $0 < \lambda < 2c'(\gamma + \rho^2)$, we get that, for all $\gamma > 0$, the following estimate holds:

$$\left(c'(\gamma + \rho^2) - \frac{\lambda}{2}\right) \|W_2^{\varepsilon +}\|^2 + \left\langle \mathcal{S}_l^+ W_2^{\varepsilon +}|_{x=0}, W_2^{\varepsilon +}|_{x=0} \right\rangle \leq \frac{2}{\lambda} \|Re \mathcal{S}^+ \widetilde{G}^+\|^2.$$

Therefore, there are $c_1 > 0$ and $C_1 > 0$ such that:

$$c_1(\gamma + \rho^2)\|W_2^{\varepsilon+}\|^2 + \left\langle \mathcal{S}_l^+ W_2^{\varepsilon+}|_{x=0}, W_2^{\varepsilon+}|_{x=0} \right\rangle \leq \frac{C_1}{\gamma + \rho^2}\|Re\mathcal{S}^+\widetilde{G}^+\|^2.$$

Adopting symmetric notations for $\underline{W}_2^{\varepsilon-}$ and adding the two estimates gives that there are $c > 0$ and $C > 0$, such that, for all $\gamma > 0$, there holds:

$$c(\gamma+\rho^2)\|W_2^{\varepsilon}\|_{L^2(\mathbb{R})}^2 + \left\langle \left(\mathcal{S}_l^+ + \mathcal{S}_l^-\right) \underline{q}_2, \underline{q}_2 \right\rangle \leq \frac{C}{\gamma + \rho^2} \left(\|Re\mathcal{S}^+\widetilde{G}^+\|^2 + \|Re\mathcal{S}^-\widetilde{G}^-\|^2 \right).$$

Proposition 3.2.15. *For all $\underline{q} \in \mathbb{C}^{2N}$, there is $\delta > 0$, $\delta' > 0$ and a set of two symmetrizers $\mathcal{S}_l^+$ and $\mathcal{S}_l^-$ such that:*

$$\left\langle \left(\mathcal{S}_l^+ + \mathcal{S}_l^-\right) \underline{q}, \underline{q} \right\rangle \geq \min(\rho\delta', \delta) \left\langle \underline{q}, \underline{q} \right\rangle.$$

Proof. Denote by q_H the N first coordinates of $\underline{q}$ and by q_P the N last ones. We have then:

$$\left\langle \left(\mathcal{S}_l^+ + \mathcal{S}_l^-\right) \underline{q}, \underline{q} \right\rangle = \rho \left\langle \left(\check{\mathcal{S}}_H^+ + \check{\mathcal{S}}_H^-\right) q_H, q_H \right\rangle + \left\langle \left(\mathcal{S}_P^+ + \mathcal{S}_P^-\right) q_P, q_P \right\rangle$$

The uniform Evans condition being satisfied, we get immediately the analogous of Proposition 3.2.10 for the parabolic part: there are two symmetrizers $\mathcal{S}_P^+$, $\mathcal{S}_P^-$ and a positive constant δ such that for all $q_P \in \mathbb{C}^N$, there holds:

$$\left\langle \left(\mathcal{S}_P^+ + \mathcal{S}_P^-\right) q_P, q_P \right\rangle \geq \delta \left\langle q_P, q_P \right\rangle.$$

For $\rho \geq C > 0$, we obtain the same way that there is a positive constant δ' such that, for all $q_H \in \mathbb{C}^N$,

$$\rho \left\langle \left(\check{\mathcal{S}}_H^+ + \check{\mathcal{S}}_H^-\right) q_H, q_H \right\rangle \geq \delta' \left\langle q_H, q_H \right\rangle.$$

Hence, for $\rho \geq C > 0$, there holds:

$$\left\langle \left(\mathcal{S}_l^+ + \mathcal{S}_l^-\right) \underline{q}, \underline{q} \right\rangle \geq \min(\delta', \delta) \left\langle \underline{q}, \underline{q} \right\rangle.$$

This inequality is true provided that the Evans Condition holds, **even if it is not uniformly**. For $\rho \geq C > 0$, due to our stability assumption holding, we had the following decomposition of $\mathbb{C}^N$:

$$\mathbb{C}^N = \mathbb{E}_-(H^+) \bigoplus \mathbb{E}_-(H^-).$$

Remark that, for all $\rho > 0$, $\mathbb{E}_-(\check{H}^+) = \mathbb{E}_-(H^+)$ and $\mathbb{E}_+(\check{H}^-) = \mathbb{E}_+(H^-)$. Moreover we had:

$$\mathbb{C}^N = \mathbb{E}_-(H^+) \bigoplus \mathbb{E}_+(H^+) = \mathbb{E}_-(H^-) \bigoplus \mathbb{E}_+(H^-).$$

For the frequencies in a neighborhood of zero, let us prove our result. By Theorem 3.2.12 and 3.2.13, we have: $\mathbb{C}^N = \underline{\mathbb{F}}_1^- \bigoplus \underline{\mathbb{F}}_1^+$, and $\mathbb{C}^N = \underline{\mathbb{F}}_2^- \bigoplus \underline{\mathbb{F}}_2^+$. For fixed $\rho > 0$, and $(\check{\tau}, \check{\gamma})$ such that $\check{\tau}^2 + \check{\gamma}^2 = 1$ with $\check{\gamma} \geq 0$, thanks to the Evans Condition holding, we have:

$$\mathbb{C}^N = \mathbb{E}_-(\check{H}^+) \bigoplus \mathbb{E}_-(\check{H}^-).$$

As a corollary of Theorem 3.2.12 and Theorem 3.2.13, as proven in [MZ04] and [Mét04], the vector bundles $\mathbb{E}_-(\check{H}^+)(\check{\zeta}, \rho)$ and $\mathbb{E}_-(\check{H}^-)(\check{\zeta}, \rho)$, defined for $\check{\zeta}$ such that $|\check{\zeta}| = 1$, with $\check{\gamma} \geq 0$ and $\rho > 0$, extends continuously to $\rho = 0$. As a matter of fact, these continuous extensions are the previously introduced linear subspaces $\underline{\mathbb{F}}_1^-$ and $\underline{\mathbb{F}}_2^-$. Since the Evans Condition holds **uniformly**, and the extensions of $\mathbb{E}_-(\check{H}^+)$ to $\underline{\mathbb{F}}_1^-$ and of $\mathbb{E}_-(\check{H}^-)$ to $\underline{\mathbb{F}}_2^-$ are continuous, we have then:

$$\underline{\mathbb{F}}_1^- \bigcap \underline{\mathbb{F}}_2^- = \{0\},$$

and therefore $\underline{\mathbb{F}}_1^- \bigoplus \underline{\mathbb{F}}_2^- = \mathbb{C}^N$

As a result, for all $q_H \in \mathbb{C}^N$, either $q_H = 0$, or $\underline{\Pi}_1^+ q_H \neq 0$ or $\underline{\Pi}_2^+ q_H \neq 0$. Moreover, by construction of $\check{\mathcal{S}}_H^{\pm}$:

$$\left\langle \check{\mathcal{S}}_H^+(\check{\zeta}, \rho) q_H, q_H \right\rangle \geq \kappa_1 |\underline{\Pi}_1^+ q_H|^2 - |\underline{\Pi}_1^- q_H|^2,$$

$$\left\langle \check{\mathcal{S}}_H^-(\check{\zeta}, \rho) q_H, q_H \right\rangle \geq \kappa_2 |\underline{\Pi}_2^+ q_H|^2 - |\underline{\Pi}_2^- q_H|^2.$$

For $q_H = 0$, the awaited inequality trivially holds. If it is not the case, since either $\underline{\Pi}_1^+ q_H \neq 0$ or $\underline{\Pi}_2^+ q_H \neq 0$, we obtain the desired result by choosing the two positive parameters κ_1 and κ_2 large enough. □

We get then the following estimate:

Proposition 3.2.16. *There are $\delta > 0$, $c > 0$ and $C > 0$ such that, for all nonzero frequencies, there holds:*

$$c(\gamma + \rho^2) \|W_2^\varepsilon\|_{L^2(\mathbb{R})}^2 + \delta \rho |W_2^\varepsilon|_{x=0}|^2 \leq \frac{C}{\gamma + \rho^2} \|G\|_{L^2(\mathbb{R})}^2. \tag{3.2.15}$$

Note that this estimate needs that either $\gamma > 0$ or $\rho > 0$ to properly control our error. This shows the need to introduce the weight $e^{-\gamma t}$ with $\gamma > 0$.

The main error estimate

In the previous chapters, we have obtained three energy estimates, each concerning a different regime of frequencies. We recall that the frequencies were respectively divided in $\widetilde{\zeta} < 1$ for the low frequencies, $1 \leq \widetilde{\zeta} \leq 2$ for the medium frequencies and $\widetilde{\zeta} > 2$ for the high frequencies. In a first step, we will rewrite our estimates (all the positive constants will be take equal to one) for the different regimes of frequencies, this time for the original variables x and ζ instead of z and $\widetilde{\zeta}$. To begin with, let us redefine here the notations $\|.\|$ and $|.|$ as follows:

$$\|f(\tau,x)\|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle f(\tau,x), f(\tau,x)\rangle \; dx \; d\tau$$

and

$$|f(\tau)|^2 = \int_{-\infty}^{\infty} \langle f(\tau), f(\tau)\rangle d\tau.$$

We will integrate the previous estimations between $-\infty$ and ∞ with respect to τ. There is $C_m > 0$ such that, for all $1 \leq |\varepsilon\zeta| \leq 2$, the energy estimate writes:

$$\|\hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon^2\|\partial_x \hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})} + |\hat{w}^\varepsilon|_{x=0}|^2 + \varepsilon^2|\partial_x \hat{w}^\varepsilon|_{x=0}|^2 \leq C_m \varepsilon^{2M}$$

There is $C_h > 0$ such that, for all $|\varepsilon\zeta| > 2$, the following estimate holds:

$$\begin{aligned}&(1+\varepsilon\tau^2+\varepsilon\gamma^2)\|\hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon^2\|\partial_x \hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})}\\ &+(1+\varepsilon\tau^2+\varepsilon\gamma^2)|\hat{w}^\varepsilon|_{x=0}|^2 + \varepsilon^2|\partial_x \hat{w}^\varepsilon|_{x=0}|^2 \leq C_h \varepsilon^{2M}.\end{aligned}$$

There is $C_l > 0$ such that, for all $|\varepsilon\zeta| < 1$, there holds:

$$(\varepsilon\gamma+\varepsilon^2\rho^2)\left(\|\hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon^2\|\partial_x \hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})}\right) + \varepsilon\rho\left(|\hat{w}^\varepsilon|_{x=0}|^2 + \varepsilon^2|\partial_x \hat{w}^\varepsilon|_{x=0}|^2\right) \leq \frac{C_l}{\varepsilon\gamma+\varepsilon^2\rho^2}\varepsilon^{2M},$$

and thus:

(3.2.16)

$$(\gamma+\varepsilon\rho^2)\left(\|\hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})} + \varepsilon\|\partial_x \hat{w}^\varepsilon\|^2_{L^2(\mathbb{R})}\right) + \rho\left(|\hat{w}^\varepsilon|_{x=0}|^2 + \varepsilon|\partial_x \hat{w}^\varepsilon|_{x=0}|^2\right) \leq \frac{C_l}{\gamma}\varepsilon^{2M-2}.$$

Note that the estimates we proved for low frequencies were for the unknown $\widetilde{W}_2^\varepsilon$. We explain here briefly how to come back to estimates on $\widetilde{W}^\varepsilon$. $\widetilde{W}_2^{\varepsilon\pm}$ are deduced from $\widetilde{W}^\varepsilon$ by a change of basis described by $\nu^\pm$. There holds:

$$\nu^\pm|_{\zeta=0} = \begin{pmatrix} Id & (A^\pm)^{-1}B \\ 0 & Id \end{pmatrix}$$

ν^+ and ν^- are continuous in ζ. Thus, recalling that $\widetilde{W}^{\varepsilon+} = \nu^+\widetilde{W}_2^{\varepsilon+}$ [resp $\widetilde{W}^{\varepsilon-} = \nu^+\widetilde{W}_2^{\varepsilon-}$], both $\widetilde{W}^{\varepsilon+}$ and $\widetilde{W}_2^{\varepsilon+}$ satisfy estimates with coefficients of the same scale in ε and ζ. Thus, adjusting the symmetrizers to match the constants allows to obtain the low frequency estimate (3.2.16).

We have to keep in mind ε is destined to tend towards zero while looking at our estimates.
Since $\hat{w}^\varepsilon$ is continuous through $\{x = 0\}$, $\hat{w}^\varepsilon|_{x=0}$ is well-defined. Let us write the simplified estimates, not involving the traces on the boundary: There is C positive such that, for all $0 < \varepsilon < 1$, there holds:

$$\|\hat{w}^\varepsilon\|_{L^2(\mathbb{R})} \leq \frac{C}{\gamma}\varepsilon^{M-1},$$

where γ is a fixed positive parameter.
Recalling that $\hat{w}^\varepsilon(\tau, x) := \int_{-\infty}^{\infty}[e^{-\gamma t}w^\varepsilon(t, x)]e^{-2\pi i\tau t}\, dt$, and using Plancherel's Theorem, we get the following result: there is C positive independent of ε and γ, such that for all function w smooth with compact support satisfying our error equation, there holds:

$$\|e^{-\gamma t}w^\varepsilon\|_{L^2((0,T)\times\mathbb{R})} \leq \frac{C}{\gamma}\varepsilon^{M-1}.$$

Therefore, since γ is a positive parameter, by constructing our approximate solution at an order $M \geq 2$, we obtain the following stability result:

Theorem 3.2.17. *There is $C > 0$ such that, for all $0 < \varepsilon < 1$:*

$$\|w^\varepsilon\|_{L^2((0,T)\times\mathbb{R})} \leq C\varepsilon.$$

3.2.5 Proof of the Uniform Lopatinski condition holding for the mixed hyperbolic problem (3.2.4).

We will now prove, by a detailed analysis of the Evans condition for low frequency, that the Uniform Lopatinski condition holds for (3.2.4) thus proving the well-posedness of the transmission problem (3.2.7).

$$\mathbb{A}(t,y,x;\zeta) := \begin{pmatrix} 0 & Id \\ \mathcal{M}(t,y,x;\zeta) & \mathcal{A}(t,y,x;\eta) \end{pmatrix}.$$

To begin with, let us fix the values of $(t,y,x) := (t_0,y_0,x_0)$ and study the behavior of $\mathbb{A}_0(\zeta) := \mathbb{A}(t_0,y_0,x_0;\zeta)$ for $|\zeta|$ in a neighborhood of zero.

Lemma 3.2.18. *There is a nonsingular matrix $\nu(\zeta)$, smooth on a neighborhood ω_0, of 0 such that:*

$$\nu(\zeta)^{-1}\mathbb{A}_0(\zeta)\nu(\zeta) = \begin{pmatrix} H(\zeta) & 0 \\ 0 & P(\zeta) \end{pmatrix} := \mathcal{G}_0(\zeta).$$

At $\zeta = 0$, we have $P(0) = B^{-1}A_d(t_0,y_0,x_0;0)$ and $H(0) = 0$.

$$\nu(0) = \begin{pmatrix} Id & (A_d)^{-1}B(t_0,y_0,x_0;0) \\ 0 & Id \end{pmatrix} := \mathcal{G}(\zeta).$$

$H(\zeta)$ is often referred to as the hyperbolic block since it satisfies, for $\zeta \in \omega_0$:

$$H(\zeta) = A(t_0,y_0,x_0;\zeta) + \mathcal{O}(|\zeta|^2).$$

A proof of this Lemma can be found in [Mét04]. Remark that:

$$\mathbb{E}_-(\mathbb{A}_0(\zeta)) = \nu(\zeta)\mathbb{E}_-(H(\zeta)) \times \mathbb{E}_-(P(\zeta)).$$

The Uniform Evans condition writes:

$$det\left(\mathbb{E}_-(\mathbb{A}^+|_{x=0}), \mathbb{E}_+(\mathbb{A}^-|_{x=0})\right) \geq C > 0.$$

When the two linear subspaces $\mathbb{E}_-(\mathbb{A}^+|_{x=0})$ and $\mathbb{E}_+(\mathbb{A}^-|_{x=0})$ extends continuously to $\zeta \neq 0$ with $\gamma > 0$, and if we denote by $\widetilde{\mathbb{E}}_-(\mathbb{A}^+|_{x=0})$ and $\widetilde{\mathbb{E}}_+(\mathbb{A}^-|_{x=0})$ the extended spaces, the Uniform Evans Condition consists in asking, for all $\zeta \neq 0$ that:

$$\widetilde{\mathbb{E}}_-(\mathbb{A}^+|_{x=0}) \bigcap \widetilde{\mathbb{E}}_+(\mathbb{A}^-|_{x=0}) = \{0\}.$$

Such extensions do exist in our case. Indeed in [MZ04], Métivier and Zumbrun proves that, under our assumptions, the following result holds, as a direct consequence of the construction of a Kreiss-type Symmetrizer. Let us denote $\rho = |\zeta|$, we have then that $\zeta = \rho\check{\zeta}$. We have then the following result:

Theorem 3.2.19. *The linear bundle $\check{\mathbb{E}}(t, y, \check{\zeta}, \rho) := \mathbb{E}_-(t, y, \rho\check{\zeta})$ has a continuous extension to $\rho = 0$, $\check{\gamma} \geq 0$.*

The Uniform Evans being satisfied for low frequencies, it implies that:

$$\left|det\left(\nu^+\left(\widetilde{\mathbb{E}}_-(A^+|_{x=0}) \times \mathbb{E}_-(P^+|_{x=0})\right), \nu^-\left(\widetilde{\mathbb{E}}_+(A^-|_{x=0}) \times \mathbb{E}_+(P^-|_{x=0})\right)\right)\right| \geq C > 0.$$

where, for $|\zeta|$ in a neighborhood of zero:

$$\nu^{\pm}(t, y; \zeta) = \begin{pmatrix} Id & (A_d^{\pm})^{-1}B_{d,d}(t, y, 0; \zeta) \\ 0 & Id \end{pmatrix} + \mathcal{O}(|\zeta|).$$

Let $\widetilde{D}_0$ denote the following determinant:

$$\left|det\left(\nu^+|_{\zeta=0}\left(\widetilde{\mathbb{E}}_-(A^+|_{x=0}) \times \mathbb{E}_-(P^+|_{x=0,\zeta=0})\right), \nu^-|_{\zeta=0}\left(\widetilde{\mathbb{E}}_+(A^-|_{x=0}) \times \mathbb{E}_+(P^-|_{x=0,\zeta=0})\right)\right)\right|$$

There is $\rho_0 > 0$ such that for all ζ such that $|\zeta| = \rho_0$, there holds:

$$\widetilde{D}_0 \geq C > 0.$$

where $P^{\pm}|_{\zeta=0} = B_{d,d}^{-1}A_d^{\pm}$.
$\nu^+|_{\zeta=0}\left(\widetilde{\mathbb{E}}_-(A^+|_{x=0}) \times \mathbb{E}_-(P^+|_{x=0,\zeta=0})\right)$ is the linear subset composed of the $\begin{pmatrix} u^+ \\ v^+ \end{pmatrix}$ such that there are $u'^+ \in \widetilde{\mathbb{E}}_-(A^+|_{x=0})$ and $v'^+ \in \mathbb{E}_-(B_{d,d}^{-1}A_d^+)$ such that :

$$\begin{pmatrix} u^+ \\ v^+ \end{pmatrix} = \begin{pmatrix} u'^+ + \left(A_d^+\right)^{-1} B_{d,d}|_{x=0}v'^+ \\ v'^+ \end{pmatrix}$$

The same way, $\nu^-|_{\zeta=0}\left(\widetilde{\mathbb{E}}_+(A^-|_{x=0}) \times \mathbb{E}_+(P^-|_{x=0,\zeta=0})\right)$ is the linear subset composed of the $\begin{pmatrix} u^- \\ v^- \end{pmatrix}$ such that there are $u'^- \in \widetilde{\mathbb{E}}_+(A^-|_{x=0})$

and
$v'^{-} \in \mathbb{E}_{+}(B_{d,d}^{-1}A_d^{-})$ such that :

$$\begin{pmatrix} u^{-} \\ v^{-} \end{pmatrix} = \begin{pmatrix} u'^{-} + \left(A_d^{-}\right)^{-1} B_{d,d}|_{x=0} v'^{-} \\ v'^{-} \end{pmatrix}$$

The low frequency Evans Condition rewrites then:

$$\nu^{-}|_{\zeta=0}\left(\widetilde{\mathbb{E}}_{+}(A^{-}|_{x=0}) \times \mathbb{E}_{+}(P^{-}|_{x=0,\zeta=0})\right) \bigcap \nu^{+}|_{\zeta=0}\left(\widetilde{\mathbb{E}}_{-}(A^{+}|_{x=0}) \times \mathbb{E}_{-}(P^{+}|_{x=0,\zeta=0})\right) = \{0\},$$

which is equivalent to the following property:
if there is $\lambda \in \mathbb{C} - \{0\}$ such that

$$\begin{cases} u'^{+} + (G_d^{+})^{-1} v'^{+} = \lambda\left(u'^{-} + (G_d^{-})^{-1} v'^{-}\right) \\ v'^{+} = \lambda v'^{-} \end{cases}$$

with $u'^{+} \in \widetilde{\mathbb{E}}_{-}(A^{+}|_{x=0})$, $v'^{+} \in \mathbb{E}_{-}(B_{d,d}^{-1}A_d^{+})$, $u'^{-} \in \widetilde{\mathbb{E}}_{+}(A^{-}|_{x=0})$ and $v'^{-} \in \mathbb{E}_{+}(B_{d,d}^{-1}A_d^{-})$, then this implies that $u'^{+} = u'^{-} = v'^{+} = v'^{-} = 0$.
Easy algebraic considerations proves this is true iff:

$$((G_d^{+})^{-1} - (G_d^{-})^{-1})\left(\mathbb{E}_{-}(G_d^{+}) \bigcap \mathbb{E}_{+}(G_d^{-})\right) \bigoplus \widetilde{\mathbb{E}}_{-}(A^{+}|_{x=0}) \bigoplus \widetilde{\mathbb{E}}_{+}(A^{-}|_{x=0}) = \mathbb{C}^N.$$

We recall Σ denotes the space:

$$\Sigma = ((B_{d,d}^{-1}A_d^{+})^{-1} - (B_{d,d}^{-1}A_d^{-})^{-1})\left(\mathbb{E}_{-}(B_{d,d}^{-1}A_d^{+}|_{x=0}) \bigcap \mathbb{E}_{+}(B_{d,d}^{-1}A_d^{-}|_{x=0})\right).$$

Thus for all $\zeta \neq 0$, such that $|\zeta| < \rho_0$, we have:

$$\widetilde{\mathbb{E}}_{-}(A^{+}|_{x=0}) \bigoplus \widetilde{\mathbb{E}}_{+}(A^{-}|_{x=0}) \bigoplus \Sigma = \mathbb{C}^N.$$

Since both of the tangential symbols $A^{+}|_{x=0}$ and $A^{-}|_{x=0}$ are homogeneous of order zero in ζ, this is equivalent to say that for all $\zeta \neq 0$, there holds:

$$\widetilde{\mathbb{E}}_{-}(A^{+}|_{x=0}) \bigoplus \widetilde{\mathbb{E}}_{+}(A^{-}|_{x=0}) \bigoplus \Sigma = \mathbb{C}^N,$$

which is an equivalent expression of the Uniform Lopatinski Condition for the mixed hyperbolic problem (3.2.4). Due to the hyperbolicity assumption, we get moreover that $\dim \mathbb{E}_{-}(A^{+}|_{x=0}) = \dim \mathbb{E}_{+}(A_d^{+}|_{x=0})$ and $\dim \mathbb{E}_{+}(A^{-}|_{x=0}) = \dim \mathbb{E}_{-}(A_d^{-}|_{x=0})$.

Remark 3.2.20. *In the 1-D framework, the Uniform Lopatinski Condition writes:*

$$\mathbb{E}_+(A_d^+)\bigoplus\mathbb{E}_-(A_d^-)\bigoplus\Sigma=\mathbb{R}^N.$$

The role of our transversality Assumption, alongside the other structure assumptions, is to guarantee Σ has the suitable dimension. This Assumption is thus crucial since, if Σ has not the right dimension, the limiting mixed hyperbolic problem has no chance of satisfying a Lopatinski Condition even though its parabolic perturbation satisfies a Uniform Evans Condition.

3.3 An open scalar question: the scalar expansive case.

For scalar hyperbolic problems of conservation laws with discontinuous coefficients, we saw in [For07c] that the expansive case was quite special to treat. This section is devoted to the open analogous nonconservative problem. To begin with, let us detail the current problematic: we have in mind to give a sense to the Cauchy problem for the hyperbolic operator $\mathcal{H}=\partial_t u+a(x)\partial_x u$ where a is piecewise constant, equal to a^+ on $\{x>0\}$ and equal to a^- on $\{x<0\}$, with $a^+>0$ and $a^-<0$:

$$\begin{cases}\partial_t u+a(x)\partial_x u=f, & x\in\mathbb{R},\\ u|_{t=0}=h \quad ,\end{cases}$$

where $f\in C_0^\infty((0,T)\times\mathbb{R})$ and $h\in C_0^\infty(\mathbb{R})$. By opting for a viscous approach, we will see that a solution of the above problem can be obtained in the vanishing viscosity limit. Moreover, our viscous approach successfully select a unique solution. Our main result is stated in Theorem 3.3.2.

Let us now describe our approach. We consider the viscous hyperbolic-parabolic problem:

$$\begin{cases}\partial_t u^\varepsilon+a(x)\partial_x u^\varepsilon-\varepsilon\partial_x^2 u^\varepsilon=f, & x\in\mathbb{R},\\ u^\varepsilon|_{t=0}=h \quad .\end{cases}\tag{3.3.1}$$

The stability of problem (3.3.1) has to be established via Kreiss-type Symmetrizers, thus explaining that we assume the coefficient to be piecewise constant in order to avoid the use of pseudodifferential calculus. We prove then a convergence result in $L^2((0,T)\times\mathbb{R})$, stating that the solution u^ε of (3.3.1) tends towards $\underline{u}$, deduced from an asymptotic analysis of the problem. More precisely, $\underline{u}$ is given by $\underline{u} := u_R\mathbf{1}_{x\geq 0} + u_L\mathbf{1}_{x<0}$, where (u_R, u_L) is the unique solution of the following problem:

$$\begin{cases} \partial_t u_R + a_R\partial_x u_R = f_R, \qquad \{x>0\}, \\ \partial_t u_L + a_L\partial_x u_L = f_L, \qquad \{x<0\}, \\ u_R|_{x=0} - u_L|_{x=0} = 0, \\ \partial_x u_R|_{x=0} - \partial_x u_L|_{x=0} = 0, \quad \forall t\in(0,T), \\ u_R|_{t=0} = h_R, u_L|_{t=0} = h_L \quad , \end{cases}$$

with f_R [resp h_R] denoting the restriction of f [resp h] to $\{x>0\}$, and f_L [resp h_L] denoting the restriction of f [resp h] to $\{x<0\}$. Note well that $\underline{u}$, deduced from this unusual, although well-posed, problem belongs to $C^0((0,T)\times\mathbb{R})\bigcap L^2((0,T)\times\mathbb{R})$. Indeed, as we shall prove below, the restriction of $\underline{u}$ to the side $\{x<0\}$ is given by:

$$\begin{cases} \partial_t u_L + a_L\partial_x u_L = f_L, \qquad \{x<0\}, \\ u_L|_{x=0} = h_L(0) + \displaystyle\int_0^t f|_{x=0}(s)\ ds, \quad \forall t\in(0,T). \\ u_L|_{t=0} = h_L \quad . \end{cases}$$

and the restriction of $\underline{u}$ to the side $\{x>0\}$ satisfies:

$$\begin{cases} \partial_t u_R + a_R\partial_x u_R = f_R, \qquad \{x>0\}, \\ u_R|_{x=0} = h_R(0) + \displaystyle\int_0^t f|_{x=0}(s)\ ds, \quad \forall t\in(0,T), \\ u_R|_{t=0} = h_R \quad . \end{cases}$$

Remark that, in general, the corner compatibilities are not satisfied here, and that $u\notin C([0,T]:H^s(\mathbb{R}))\ \forall s>\frac{3}{2}$ for example, even though the datas f and h are smooth.

Remark 3.3.1. *$\underline{u}$ is also given by:*

$$\underline{u}(t,x) = h(x) + \int_0^t \underline{v}(s,x)\ ds,$$

where $\underline{v} := v_L \mathbf{1}_{x<0} + v_R \mathbf{1}_{x\geq 0}$ *is the solution of the well-posed classical transmission problem:*

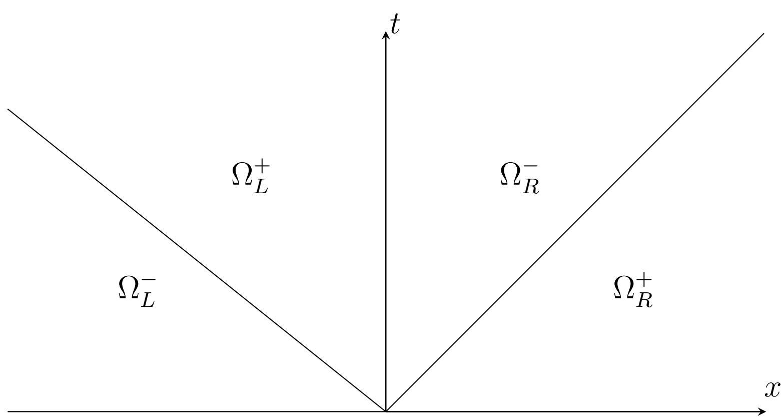
$$\begin{cases} \partial_t v_R + a_R \partial_x v_R = \partial_t f_R, & \{x > 0\}, \\ \partial_t v_L + a_L \partial_x v_L = \partial_t f_L, & \{x < 0\}, \\ \dfrac{1}{a_R} v_R|_{x=0} - \dfrac{1}{a_L} v_L|_{x=0} = \dfrac{1}{a_R} \partial_t f_R|_{x=0} - \dfrac{1}{a_L} \partial_t f_L|_{x=0}, \\ v_R|_{x=0} - v_L|_{x=0} = 0, \\ v_R|_{t=0} = f_R - a_R \partial_x h_R, \quad v_L|_{t=0} = f_L - a_L \partial_x h_L \quad . \end{cases}$$

This problem is labeled as classical since it is equivalent to a mixed hyperbolic problem satisfying a Uniform Lopatinski Condition.

As an illustration, let us compute $\underline{u}$ in the case where $f = 0$. We will first introduce some notations. We denote for instance:

$$\Omega_L^+ = \{(t, x) \in (0, T) \times \mathbb{R}^{*-} : x - a_L t > 0\},$$

where the 'L' stands for 'on left hand side of $\{x = 0\}$' and the $+$ is related to the sign of $x - a_L t$. We define in the same manner: Ω_L^-, Ω_R^+ and Ω_R^-.



We get that, for all $(t, x) \in \Omega_L^+ \bigcup \Omega_R^- \bigcup \{x = 0\}$,

$$\underline{u}(t, x) = h(0),$$

for all $(t, x) \in \Omega_R^+$,

$$\underline{u}(t, x) = h_R(x - a_R t),$$

and for all $(t,x) \in \Omega_L^-$,

$$\underline{u}(t,x) = h_L(x - a_L t).$$

This example shows clearly the discontinuity of $\partial_x \underline{u}$ occurring across the lines $\Gamma_R = \{(t,x) \in (0,T) \times \mathbb{R}_+^*, \quad x - a_R t = 0\}$ and $\Gamma_L = \{(t,x) \in (0,T) \times \mathbb{R}_-^*, \quad x - a_L t = 0\}$. The following Theorem is our main result:

Theorem 3.3.2. *There is $C > 0$ such that, for all $0 < \varepsilon < 1$, there holds:*

$$\|u^\varepsilon - \underline{u}\|_{L^2((0,T)\times\mathbb{R})} \leq C\varepsilon,$$

where u^ε is the solution of (3.3.1)*.*

Remark 3.3.3. *The rate of convergence obtained here is better than the one we had on the analogous conservative problem treated in [For07c]. This is directly explained by a boundary layer analysis of the two problems, which shows that, in [For07c], strong amplitude boundary layers forms, whereas in our present case, only weak amplitude boundary layers form.*

The proof of Theorem 3.3.2 is divided into two parts. First, we will construct an approximate solution of (3.3.1) at any order. Then, we will show that a Uniform Evans Condition holds for an equivalent problem, hence yielding the desired stability estimates.

3.3.1 Construction of an approximate solution.

We shall begin by constructing an approximate solution of problem (3.3.1). As a first step, we will reformulate problem (3.3.1) in an equivalent manner. The restrictions of u^ε to $\{x > 0\}$ and $\{x < 0\}$, denoted respectively by u_L^ε and u_R^ε satisfy the following transmission problem:

$$(3.3.2) \qquad \begin{cases} \partial_t u_R^\varepsilon + a_R \partial_x u_R^\varepsilon - \varepsilon \partial_x^2 u_R^\varepsilon = f_R, & \{x > 0\}, t \in (0,T), \\ \partial_t u_L^\varepsilon + a_L \partial_x u_L^\varepsilon - \varepsilon \partial_x^2 u_L^\varepsilon = f_L, & \{x < 0\}, t \in (0,T), \\ u_R^\varepsilon|_{x=0} - u_L^\varepsilon|_{x=0} = 0, \\ \partial_x u_R^\varepsilon|_{x=0} - \partial_x u_L^\varepsilon|_{x=0} = 0, \\ u_R^\varepsilon|_{t=0} = h_R, \\ u_L^\varepsilon|_{t=0} = h_L \quad . \end{cases}$$

Let us introduce $L_R^\varepsilon = \partial_t + a_R\partial_x - \varepsilon\partial_x^2$ and $L_L^\varepsilon = \partial_t + a_L\partial_x - \varepsilon\partial_x^2$. We perform the construction of the approximate solution separately on the four domains Ω_L^-, Ω_L^+, Ω_R^+ and Ω_R^-. . We will denote by $u^\varepsilon_{app,L,+}$ the restriction of u^ε_{app} to Ω_L^+ and so on. Let us present the different profiles and their ansatz:

$$u^\varepsilon_{app,L,+}(t,x) = \sum_{n=0}^{M}\left(\underline{\mathbf{U}}_{L,n,+}(t,x) + \mathbf{U}^c_{L,n,+}\left(t, \frac{x-a_Lt}{\sqrt{\varepsilon}}\right)\right)\varepsilon^{\frac{n}{2}},$$

where the profiles $\underline{\mathbf{U}}_{L,n,+}$ belongs to $H^\infty(\Omega_L^+)$ and the characteristic boundary layer profiles $\mathbf{U}^c_{L,n,+}(t,\theta_L)$ belongs to $e^{-\delta|\theta_L|}H^\infty((0,T)\times\mathbb{R}^{*+})$, for some $\delta > 0$. We will take a similar ansatz for $u^\varepsilon_{app,L,-}$, $u^\varepsilon_{app,R,-}$ and $u^\varepsilon_{app,R,+}$ over their respective domains. Let us explain the different steps of the construction of the approximate solution. We begin by constructing the underlined profiles $\underline{\mathbf{U}}_n$ in cascade; the boundary layer profiles $\mathbf{U}^c_n$ are then computed as a last step. We construct our profiles such that, for all fixed $\varepsilon > 0$, u^ε_{app} belongs to $C^1((0,T)\times\mathbb{R})$. In what follows, we will note:

$$\underline{\mathbf{U}}_{R,j}(t,x) := \underline{\mathbf{U}}_{R,j,+}(t,x)\mathbf{1}_{(t,x)\in\Omega_R^+} + \underline{\mathbf{U}}_{R,j,-}(t,x)\mathbf{1}_{(t,x)\in\Omega_R^-}.$$

Next, we will write:

$$\mathbf{U}^c_{R,j}\left(t,x,\frac{x-a_Rt}{\sqrt{\varepsilon}}\right) := \mathbf{U}^c_{R,j,+}\left(t,\frac{x-a_Rt}{\sqrt{\varepsilon}}\right)\mathbf{1}_{(t,x)\in\Omega_R^+} + \mathbf{U}^c_{R,j,-}\left(t,\frac{x-a_Rt}{\sqrt{\varepsilon}}\right)\mathbf{1}_{(t,x)\in\Omega_R^-}.$$

Note well that the dependence of $\mathbf{U}^c_{R,j}$ in x is a bit subtle. Actually, $\mathbf{U}^c_{R,j}$ is piecewise constant with respect to x on each side of Γ_R, which explains that $\mathbf{U}^c_{R,j,+}$ and $\mathbf{U}^c_{R,j,-}$ have no direct dependency in x. Due to their particular meaning, we prefer denoting the profiles $\underline{\mathbf{U}}_{R,0}$ and $\underline{\mathbf{U}}_{L,0}$ by u_R and u_L. Let us note $\mathcal{H}_R$ the differential operator

$$\mathcal{H}_R := \partial_t + a_R\partial_x$$

and $\mathcal{P}_R$ the differential operator

$$\mathcal{P}_R := \partial_t + a_R\partial_x - \partial^2_{\theta_R}.$$

We have

$$L_R^\varepsilon\, u^\varepsilon_{R,app}\left(t,x,\frac{x-a_Rt}{\sqrt{\varepsilon}}\right) = \sum_{j=0}^{M+1} L_{R,j}\left(t,x,\frac{x-a_Rt}{\sqrt{\varepsilon}}\right)\varepsilon^{\frac{j}{2}}$$

where

$$L_{R,0} = \mathcal{H}_R u_R + \mathcal{P}_R U^c_{R,0},$$

$$L_{R,1} = \mathcal{H}_R \underline{\mathbf{U}}_{R,1} + \mathcal{P}_R U^c_{R,1} - 2\partial_x \partial_{\theta_R} U^c_{R,0},$$

and, for $2 \leq j \leq M-1$, we get:

$$L_{R,j} = \mathcal{H}_R \underline{\mathbf{U}}_{R,j} + \mathcal{P}_R U^c_{R,j} - \partial_x \left(2\partial_{\theta_R} U^c_{R,j-1} + \partial_x \underline{\mathbf{U}}_{R,j-2} + \partial_x U^c_{R,j-2}\right),$$

$$L_{R,M} = \mathcal{P}_R U^c_{R,M} - \partial_x \left(2\partial_{\theta_R} U^c_{R,M-1} + \partial_x \underline{\mathbf{U}}_{R,M-2} + \partial_x U^c_{R,M-2}\right),$$

$$L_{R,M+1} = -\partial_x \left(2\partial_{\theta_R} U^c_{R,M} + \partial_x \underline{\mathbf{U}}_{R,M-1} + \partial_x U^c_{R,M-1}\right).$$

Symmetrically, there holds:

$$L^\varepsilon_L \; u^\varepsilon_{L,app}\left(t, x, \frac{x - a_L t}{\sqrt{\varepsilon}}\right) = \sum_{j=0}^{M+1} L_{L,j}\left(t, x, \frac{x - a_L t}{\sqrt{\varepsilon}}\right) \varepsilon^{\frac{j}{2}}$$

where, for instance, $L_{L,2}$ is given by:

$$L_{L,2} = \mathcal{H}_L \underline{\mathbf{U}}_{L,2} + \mathcal{P}_L U^c_{L,2} - \partial_x \left(2\partial_{\theta_L} + \partial_x u_L + \partial_x U^c_{L,0}\right),$$

with $\mathcal{H}_L$ and and $\mathcal{P}_L$ defined by:

$$\mathcal{H}_L := \partial_t + a_L \partial_x$$

$$\mathcal{P}_L := \partial_t + a_L \partial_x - \partial^2_{\theta_L}.$$

Plugging $u^\varepsilon_{L,app}$ and $u^\varepsilon_{R,app}$ in the problem (3.3.2) and identifying the terms with the same scale in ε, making then $|\theta_L|$ and $|\theta_R|$ tend to infinity, we obtain the profiles equations satisfied by the underlined profiles. Let us begin by writing the equations satisfied by $\underline{\mathbf{U}}_{L,j}$ and $\underline{\mathbf{U}}_{R,j}$ for all $0 \leq j \leq M-1$. Thanks to the transmission conditions we had on the viscous problem (3.3.2), we get:

$$\begin{cases} u_{L,+}|_{x=0} - u_{R,-}|_{x=0} = 0, \\ \partial_x u_{L,+}|_{x=0} - \partial_x u_{R,-}|_{x=0} = 0, \end{cases}$$

and thus $(u_{R,-}, u_{L,+})$ satisfies the following transmission problem:

$$\begin{cases} \partial_t u_{R,-} + a_R \partial_x u_{R,-} = f_{R,-}, & (t,x) \in \Omega^-_R, \\ \partial_t u_{L,+} + a_L \partial_x u_{L,+} = f_{L,+}, & (t,x) \in \Omega^+_L, \\ u_{L,+}|_{x=0} - u_{R,-}|_{x=0} = 0, & \\ \partial_x u_{L,+}|_{x=0} - \partial_x u_{R,-}|_{x=0} = 0 & . \end{cases} \tag{3.3.3}$$

As a result, the profile $u_{R,-}$ is the unique solution of:

$$\begin{cases} \partial_t u_{R,-} + a_R \partial_x u_{R,-} = f_{R,-}, & (t,x) \in \Omega_R^-, \\ u_{R,-}|_{x=0} = h(0) + \displaystyle\int_0^t f|_{x=0}(s)\ ds & , \end{cases}$$

and the profile $u_{L,+}$ is given by:

$$\begin{cases} \partial_t u_{L,+} + a_L \partial_x u_{L,+} = f_{L,+}, & (t,x) \in \Omega_L^+, \\ u_{L,+}|_{x=0} = h(0) + \displaystyle\int_0^t f|_{x=0}(s)\ ds & . \end{cases}$$

Proof. The first boundary condition of (3.3.3) gives: $\partial_t u_{L,+}|_{x=0} = \partial_t u_{R,-}|_{x=0}$. Using then the equation, we obtain:

$$\partial_x u_{R,-}|_{x=0} = \frac{1}{a_R} \left(f_{R,-}|_{x=0} - \partial_t u_{R,-}|_{x=0} \right),$$

and

$$\partial_x u_{L,+}|_{x=0} = \frac{1}{a_L} \left(f_{L,+}|_{x=0} - \partial_t u_{L,+}|_{x=0} \right).$$

Using the second boundary condition, we have thus

$$a_L \left(f|_{x=0} - \partial_t u_{R,-}|_{x=0} \right) = a_R \left(f|_{x=0} - \partial_t u_{L,+}|_{x=0} \right),$$

therefore

$$\partial_t u_{L,+}|_{x=0} = \partial_t u_{R,-}|_{x=0} = f|_{x=0}.$$

Hence, there holds:

$$u_{L,+}|_{x=0} = u_{R,-}|_{x=0} = h(0) + \int_0^t f|_{x=0}(s)\ ds.$$

□

Moreover, as we could have forecasted, the profiles $u_{R,+}$ and $u_{L,-}$ satisfy the following well-posed hyperbolic problems:

$$\begin{cases} \partial_t u_{R,+} + a_R \partial_x u_{R,+} = f_{R,+}, & (t,x) \in \Omega_R^+, \\ u_R|_{t=0} = h_R & , \end{cases}$$

$$\begin{cases} \partial_t u_{L,-} + a_L \partial_x u_{L,-} = f_{L,-}, & (t,x) \in \Omega_L^-, \\ u_L|_{t=0} = h_L & . \end{cases}$$

Since these equations are well-posed, the function $\underline{u}$ is now perfectly defined. Let us go on with the construction of the next profiles. $\underline{\mathbf{U}}_{R,1}$ and $\underline{\mathbf{U}}_{L,1}$ are given by:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{R,1,-} + a_R \partial_x \underline{\mathbf{U}}_{R,1,-} = 0, & (t,x) \in \Omega_R^-, \\ \partial_t \underline{\mathbf{U}}_{L,1,+} + a_L \partial_x \underline{\mathbf{U}}_{L,1,+} = 0, & (t,x) \in \Omega_L^+, \\ \underline{\mathbf{U}}_{L,1,+}|_{x=0} = \underline{\mathbf{U}}_{R,1,-}|_{x=0} = 0 & . \end{cases}$$

Thus $\underline{\mathbf{U}}_{L,1,+} = 0$ and $\underline{\mathbf{U}}_{R,1,-} = 0$.

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{R,1,+} + a_R \partial_x \underline{\mathbf{U}}_{R,1,+} = 0, & (t,x) \in \Omega_R^+, \\ \underline{\mathbf{U}}_{R,1,+}|_{t=0} = 0 & , \end{cases}$$

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{L,1,-} + a_L \partial_x \underline{\mathbf{U}}_{L,1,-} = 0, & (t,x) \in \Omega_L^-, \\ \underline{\mathbf{U}}_{L,1,-}|_{t=0} = 0 & . \end{cases}$$

Hence $\underline{\mathbf{U}}_{R,1,+} = 0$ and $\underline{\mathbf{U}}_{L,1,-} = 0$. Actually, we see by induction that for all $n \in \mathbb{N}$, we have $\underline{U}^{\pm}_{R,2n+1,\pm} = 0$ and $\underline{U}_{L,2n+1,\pm} = 0$. On the other hand for $n \in \mathbb{N}^*$, the profiles $\underline{U}_{L,2n,\pm}$ and $\underline{U}_{R,2n,\pm}$ are given by the following well-posed hyperbolic problems. The first equation we get is:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{R,2n,-} + a_R \partial_x \underline{\mathbf{U}}_{R,2n,-} = \partial_x^2 \underline{\mathbf{U}}_{R,2n-2,-}, & (t,x) \in \Omega_R^-, \\ \partial_t \underline{\mathbf{U}}_{L,2n,+} + a_L \partial_x \underline{\mathbf{U}}_{L,2n,+} = \partial_x^2 \underline{\mathbf{U}}_{L,2n-2,+}, & (t,x) \in \Omega_L^+, \\ \underline{\mathbf{U}}_{R,2n,-}|_{x=0} - \underline{\mathbf{U}}_{L,2n,+}|_{x=0} = 0, & \\ \partial_x \underline{\mathbf{U}}_{R,2n,-}|_{x=0} - \partial_x \underline{\mathbf{U}}_{L,2n,+}|_{x=0} = 0, & \\ \underline{\mathbf{U}}_{R,2n,-}|_{t=0} = 0, \quad \underline{\mathbf{U}}_{L,2n,+}|_{t=0} = 0 & . \end{cases}$$

The same way as before, we obtain that $\underline{\mathbf{U}}_{R,2n,-}$ and $\underline{\mathbf{U}}_{L,2n,+}$ are the solutions of the following well-posed hyperbolic problems:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{R,2n,-} + a_R \partial_x \underline{\mathbf{U}}_{R,2n,-} = \partial_x^2 \underline{\mathbf{U}}_{R,2n-2,-}, & (t,x) \in \Omega_R^-, \\ \underline{\mathbf{U}}_{R,2n,-}|_{x=0} = \displaystyle\int_0^t \partial_x^2 \underline{\mathbf{U}}_{R,2n-2,-}|_{x=0}(s)\, ds & , \end{cases}$$

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{L,2n,+} + a_L \partial_x \underline{\mathbf{U}}_{L,2n,+} = \partial_x^2 \underline{\mathbf{U}}_{L,2n-2,+}, & (t,x) \in \Omega_L^+, \\ \underline{\mathbf{U}}_{L,2n,+}|_{x=0} = \displaystyle\int_0^t \partial_x^2 \underline{\mathbf{U}}_{L,2n-2,+}|_{x=0}(s)\, ds & . \end{cases}$$

Moreover, there holds:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{R,2n,+} + a_R \partial_x \underline{\mathbf{U}}_{R,2n,+} = \partial_x^2 \underline{\mathbf{U}}_{R,2n-2,+}, & (t,x) \in \Omega_R^+, \\ \underline{\mathbf{U}}_{R,2n,+}|_{t=0} = 0 \quad , \end{cases}$$

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{L,2n,-} + a_L \partial_x \underline{\mathbf{U}}_{L,2n,-} = \partial_x^2 \underline{\mathbf{U}}_{L,2n-2,-}, & (t,x) \in \Omega_L^-, \\ \underline{\mathbf{U}}_{L,2n,-}|_{t=0} = 0 \quad . \end{cases}$$

In conclusion, all the profiles $\underline{U}_n$ are constructed by induction.

We turn now to the construction of the boundary layer profiles $U^c_{L,j,\pm}(t,\theta_L)$ and $U^c_{R,j,\pm}(t,\theta_R)$. We will use the relations imposed on the profiles by the transmission conditions: $[u^\varepsilon_{app}]_{\Gamma_R} = 0$, $[\partial_x u^\varepsilon_{app}]_{\Gamma_R} = 0$, $[u^\varepsilon_{app}]_{\Gamma_L} = 0$, and $[\partial_x u^\varepsilon_{app}]_{\Gamma_L} = 0$; $[u^\varepsilon_{app}]_{\Gamma_R}$ stands for the jump of u^ε_{app} through Γ_R defined, for all $t \in (0,T)$ by:

$$[u^\varepsilon_{app}]_{\Gamma_R}(t) := \lim_{x \to a_R t, x > a_R t} u^\varepsilon_{app}\left(t, x, \frac{x - a_R t}{\sqrt{\varepsilon}}\right) - \lim_{x \to a_R t, x < a_R t} u^\varepsilon_{app}\left(t, x, \frac{x - a_R t}{\sqrt{\varepsilon}}\right).$$

$[u^\varepsilon_{app}]_{\Gamma_L}(t)$ is defined the same way. Because u^ε_{app} belongs to $C^1((0,T) \times \mathbb{R}^*)$, for all $0 \leq j \leq M$, we have:

$$[U^c_{L,j}]_L = -[\underline{\mathbf{U}}_{L,j}]_{\Gamma_L},$$

$$[U^c_{R,j}]_R = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}.$$

Let $[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}$ be given, for all $t \in (0,T)$, by:

$$[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}(t) = \lim_{x \to a_R t, x > a_R t} \underline{\mathbf{U}}_{R,j,+}(t,x) - \lim_{x \to a_R t, x < a_R t} \underline{\mathbf{U}}_{R,j,-}(t,x)$$

and $[U^c_{R,j}]_R$ be defined, for all $t \in (0,T)$, by:

$$[U^c_{R,j}]_R(t) = \lim_{\theta_R \to 0^+} U^c_{R,j,+}(t,\theta_R) - \lim_{\theta_R \to 0^-} U^c_{R,j,-}(t,\theta_R).$$

To avoid writing the exact symmetric equations on $\{x > 0\}$ and $\{x < 0\}$, let us only proceed with the construction of the boundary layer profiles $U^c_{R,j,\pm}$. Referring to the computations above, for all $1 \leq j \leq M+1$, the following quantity must not have any Dirac measure in it:

$$\partial_x \left(\partial_{\theta_R} U^c_{R,j-1} + \frac{1}{2}(\partial_x(\underline{\mathbf{U}}_{R,j-2} + U^c_{R,j-2})) \right),$$

Our first boundary condition: $[U^c_{R,j}]_R = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}$, ensures that, even if $\partial_x(\underline{\mathbf{U}}_{R,j-2}+U^c_{R,j-2})$ is, in general, discontinuous on Γ_R, it has no Dirac Measure. $\partial_x(\partial_x(\underline{\mathbf{U}}_{R,j-2}+U^c_{R,j-2}))$ is the derivative of such a function and thus has a Dirac Measure. Let us describe this singularity: if we fix $t=t_0$, the Dirac measure forming is

$$\left([\partial_x \underline{\mathbf{U}}_{R,j-2}]|_{x=a_R t_0} + [\partial_x U^c_{R,j-2}]_R(t_0)\right)\ \delta_{x=a_R t_0},$$

where $[\omega]|_{x=a_R t_0} = \lim_{x\to a_R t_0, x>a_R t_0}\omega - \lim_{x\to a_R t_0, x<a_R t_0}\omega$.
On the other hand, if $\partial_{\theta_R} U^c_{R,j-1}$ is discontinuous through Γ_R, $\partial_x\left(\partial_{\theta_R} U^c_{R,j-1}\right)$ has a Dirac measure given, for $t=t_0$ by:

$$[\partial_{\theta_R} U^c_{R,j-1}]_R\ \delta_{x=a_R t_0}.$$

In order to ensure the sum of the two Dirac measure vanishes, the second boundary condition we get is that, $\forall t\in(0,T)$:

$$[\partial_{\theta_R} U^c_{R,j-1}]_R(t) = -\frac{1}{2}\left([\partial_x \underline{\mathbf{U}}_{R,j-2}]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-2}(t)]_R\right).$$

The profiles $U^c_{R,0,+}$ and $U^c_{R,0,-}$ are solution of the following heat equation:

$$\begin{cases} \partial_t U^c_{R,0,+} - \partial^2_{\theta_R} U^c_{R,0,+} = 0 \quad t\in(0,T),\ \{\theta_R>0\},\\ \partial_t U^c_{R,0,-} - \partial^2_{\theta_R} U^c_{R,0,-} = 0 \quad t\in(0,T),\ \{\theta_R<0\},\\ [U^c_{R,0}]_R(t) = -[u_R]_{\Gamma_R}, \quad \forall t\in(0,T),\\ [\partial_{\theta_R} U^c_{R,j}]_R(t) = 0, \quad \forall t\in(0,T),\\ U^c_{R,j,+}|_{t=0} = 0,\\ U^c_{R,j,-}|_{t=0} = 0 \quad . \end{cases}$$

Note well that, since $[u_R]_{\Gamma_R}=0$, the profiles $U^c_{R,0}$ and $U^c_{L,0}$ are both equal to zero; this shows that the characteristic boundary layers forming are of weak amplitude.
For all $1\le j\le M$, the profiles $U^c_{R,j,+}$ and $U^c_{R,j,-}$ are given by:

$$\begin{cases} \partial_t U^c_{R,j,+} - \partial^2_{\theta_R} U^c_{R,j,+} = 0 \quad t\in(0,T),\ \{\theta_R>0\},\\ \partial_t U^c_{R,j,-} - \partial^2_{\theta_R} U^c_{R,j,-} = 0 \quad t\in(0,T),\ \{\theta_R<0\},\\ [U^c_{R,j}]_R(t) = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R}, \quad \forall t\in(0,T),\\ [\partial_{\theta_R} U^c_{R,j}]_R(t) = -\frac{1}{2}\left([\partial_x \underline{\mathbf{U}}_{R,j-1}(t)]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-1}(t)]_R\right), \quad \forall t\in(0,T),\\ U^c_{R,j,+}|_{t=0} = 0,\\ U^c_{R,j,-}|_{t=0} = 0 \quad . \end{cases}$$

Let us now prove the well-posedness of these problems. We take $\psi_{R,j}$ in $H^{\infty}((0,T)\times\mathbb{R}^*)$ such that

$$[\psi_{R,j}]_R = -[\underline{\mathbf{U}}_{R,j}]_{\Gamma_R},$$

and

$$[\partial_{\theta_R}\psi_{R,j}]_R(t) = -\frac{1}{2}\left([\partial_x\underline{\mathbf{U}}_{R,j-1}(t)]_{\Gamma_R}(t) + [\partial_x U^c_{R,j-1}(t)]_R\right).$$

We can then compute $U^c_{R,j} := U^c_{R,j,+}\mathbf{1}_{\theta_R>0} + U^c_{R,j,-}\mathbf{1}_{\theta_R<0}$ by:

$$U^c_{R,j} := \psi_{R,j} + V^c_{R,j}.$$

$V^c_{R,j}$ is then the solution of the classical heat equation:

$$\begin{cases} \left(\partial_t V^c_{R,j} - \partial^2_{\theta_R}\right) V^c_{R,j} = \varphi^*_{R,j}, \quad (t,\theta_R)\in(0,T)\times\mathbb{R}, \\ V^c_{R,j}|_{t=0} = 0 \quad . \end{cases}$$

and $\varphi^*_{R,j}$ is given by:

$$\varphi^*_{R,j} := -\left(\partial_t\psi_{R,j} - \partial^2_{\theta_R}\psi_{R,j}\right).$$

The profiles can thus be constructed by induction using the scheme just introduced.

3.3.2 Stability estimates.

We will now prove stability estimates.
We define the error $w^\varepsilon := u^\varepsilon_{app} - u^\varepsilon$. Let us denote by $w^{\varepsilon\pm}$ the restriction of w^ε to $\pm x>0$. $(w^{\varepsilon+}, w^{\varepsilon-})$ is then solution of the transmission problem:

$$\begin{cases} \partial_t w^{\varepsilon+} + a_R\partial_x w^{\varepsilon+} - \varepsilon\partial_x^2 w^{\varepsilon+} = \varepsilon^M R^{\varepsilon+}, \qquad x>0, t\in(0,T), \\ \partial_t w^{\varepsilon-} + a_L\partial_x w^{\varepsilon-} - \varepsilon\partial_x^2 w^{\varepsilon-} = \varepsilon^M R^{\varepsilon-}, \qquad x<0, t\in(0,T), \\ w^{\varepsilon+}|_{x=0} - w^{\varepsilon-}|_{x=0} = 0, \\ \partial_x w^{\varepsilon+}|_{x=0} - \partial_x w^{\varepsilon-}|_{x=0} = 0, \\ w^{\varepsilon+}|_{t=0} = 0, \quad w^{\varepsilon-}|_{t=0} = 0. \end{cases}$$

By construction of our approximate solution, R^ε belongs to $L^2((0,T)\times\mathbb{R})$.

Like we have done previously for systems, we have to extend the definition of w^ε to $(t,x) \in \mathbb{R}^2$. In this paper, for the sake of simplicity, we will make a slight abuse of notations and write:

$$\begin{cases} \partial_t w^{\varepsilon+} + a_R \partial_x w^{\varepsilon+} - \varepsilon \partial_x^2 w^{\varepsilon+} = \varepsilon^M R^{\varepsilon+}, & x > 0, t \in \mathbb{R}, \\ \partial_t w^{\varepsilon-} + a_L \partial_x w^{\varepsilon-} - \varepsilon \partial_x^2 w^{\varepsilon-} = \varepsilon^M R^{\varepsilon-}, & x < 0, t \in \mathbb{R}, \\ w^{\varepsilon+}|_{x=0} - w^{\varepsilon-}|_{x=0} = 0, \\ \partial_x w^{\varepsilon+}|_{x=0} - \partial_x w^{\varepsilon-}|_{x=0} = 0, \\ w^{\varepsilon+}|_{t<0} = 0, \quad w^{\varepsilon-}|_{t<0} = 0, \end{cases}$$

with R^ε belonging to $L^2(\mathbb{R}^2)$ and vanishing in the past. We prove in [For07a], in a more general framework, that we can do so.

We will now reformulate this problem into an equivalent problem, posed on one side of the boundary. Defining $\widetilde{w}^\varepsilon := \begin{pmatrix} w^{\varepsilon+}(t,x) \\ w^{\varepsilon-}(t,-x) \end{pmatrix}$, the error equation rewrites as the doubled problem on one side of the boundary:

$$\begin{cases} \widetilde{\mathcal{H}}^\varepsilon \widetilde{w}^\varepsilon = \varepsilon^M \widetilde{R}^\varepsilon, \quad \{x > 0\}, \\ \Gamma \widetilde{w}^\varepsilon|_{x=0} = 0, \\ \widetilde{w}^\varepsilon|_{t<0} = 0. \end{cases}$$

where $\mathcal{H}^\varepsilon = \partial_t + \widetilde{A}\partial_x - \varepsilon \partial_x^2$,

$$\widetilde{A} = \begin{bmatrix} a_R & 0 \\ 0 & -a_L \end{bmatrix}, \text{ and } \Gamma = \begin{bmatrix} 1 & -1 \\ \partial_x & \partial_x \end{bmatrix}.$$

Let us admit for now the following Proposition that will be proved in the next section.

Proposition 3.3.4. $(\widetilde{\mathcal{H}}^\varepsilon, \Gamma)$ *satisfies a Uniform Evans Condition.*

As established earlier in the paper, if our linear mixed parabolic problem satisfies a Uniform Evans Condition, the following stability estimate holds:

$$\|u^\varepsilon - u^\varepsilon_{app}\|_{L^2((0,T)\times\mathbb{R})} = \mathcal{O}(\varepsilon^{\frac{M-1}{2}}),$$

taking M large enough achieves then the proof of Theorem 3.3.2.

3.4 Proof of Proposition 3.3.4.

In this section we will prefer using the notations a^+ and a^- instead of a_R and a_L. We refer to [For07a] for computations of the Evans function for 2×2 systems. In our present case, we have:

$$\mathbb{A}^{\pm}(\widetilde{\zeta}) = \begin{pmatrix} 0 & 1 \\ i\widetilde{\tau} + \widetilde{\gamma} & a^{\pm} \end{pmatrix}$$

3.4.1 Computation of the Evans function for medium frequencies.

There holds:

$$\mathbb{E}_-(\mathbb{A}^+(\widetilde{\zeta})) = Span\left\{ \begin{pmatrix} 1 \\ \mu_-^+(\widetilde{\zeta}) \end{pmatrix} \right\}$$

where μ_-^+ denotes the eigenvalue of $\mathbb{A}^+$ with negative real part and is given by:

$$\mu_-^+(\widetilde{\zeta}) = \frac{1}{2}a^+ - \frac{1}{4}\left(((a^+)^2 + 4\widetilde{\gamma})^2 + 16\widetilde{\tau}^2\right)^{\frac{1}{4}} \left(\left(1 + \frac{16\widetilde{\tau}^2}{((a^+)^2 + 4\widetilde{\gamma})^2}\right)^{-\frac{1}{2}} + 1\right)$$

$$-i\ sign(\widetilde{\tau})\frac{1}{4}\left(((a^+)^2 + 4\widetilde{\gamma})^2 + 16\widetilde{\tau}^2\right)^{\frac{1}{4}} \left(1 - \left(1 + \frac{16\widetilde{\tau}^2}{((a^+)^2 + 4\widetilde{\gamma})^2}\right)^{-\frac{1}{2}}\right)$$

Moreover, we have:

$$\mathbb{E}_+(\mathbb{A}^-(\widetilde{\zeta})) = Span\left\{ \begin{pmatrix} 1 \\ \mu_+^-(\widetilde{\zeta}) \end{pmatrix} \right\}$$

where μ_+^- denotes the eigenvalue of $\mathbb{A}^-$ with positive real part and is given by:

$$\mu_+^-(\widetilde{\zeta}) = \frac{1}{2}a^- + \frac{1}{4}\left(((a^-)^2 + 4\widetilde{\gamma})^2 + 16\widetilde{\tau}^2\right)^{\frac{1}{4}} \left(\left(1 + \frac{16\widetilde{\tau}^2}{((a^-)^2 + 4\widetilde{\gamma})^2}\right)^{-\frac{1}{2}} + 1\right)$$

$$+i\ sign(\widetilde{\tau})\frac{1}{4}\left(((a^-)^2 + 4\widetilde{\gamma})^2 + 16\widetilde{\tau}^2\right)^{\frac{1}{4}} \left(1 - \left(1 + \frac{16\widetilde{\tau}^2}{((a^-)^2 + 4\widetilde{\gamma})^2}\right)^{-\frac{1}{2}}\right)$$

If we consider $\widetilde{\zeta}$ such that $0 < c \leq |\widetilde{\zeta}| \leq C < \infty$, an Evans function is the modulus of the following determinant:

$$\begin{vmatrix} 1 & 1 \\ \mu_-^+(\widetilde{\zeta}) & \mu_+^-(\widetilde{\zeta}) \end{vmatrix}$$

that is to say: $|\mu_+^-(\widetilde{\zeta}) - \mu_-^+(\widetilde{\zeta})|$, since μ_+^- keeps a positive real part and μ_-^+ keeps a negative real part, for all $\widetilde{\zeta}$ such that $0 < c \leq |\widetilde{\zeta}| \leq C < \infty$, there holds:

$$\left|\mu_+^-(\widetilde{\zeta}) - \mu_-^+(\widetilde{\zeta})\right| > 0.$$

Hence the Evans Condition is checked for medium frequencies.

3.4.2 Computation of the asymptotic Evans function when $|\widetilde{\zeta}| \to \infty$.

Λ is defined by:

$$\Lambda(\widetilde{\zeta}) = \left(1 + \widetilde{\tau}^2 + \widetilde{\gamma}^2\right)^{\frac{1}{2}}$$

We recall that the scaled eigenspaces for high frequencies write then:

$$\mathbb{E}_-(\mathbb{A}^+(\widetilde{\zeta})) = Span\left\{\begin{pmatrix} 1 \\ \Lambda^{-1}\mu_-^+(\widetilde{\zeta}) \end{pmatrix}\right\}$$

$$\mathbb{E}_+(\mathbb{A}^-(\widetilde{\zeta})) = Span\left\{\begin{pmatrix} 1 \\ \Lambda^{-1}\mu_+^-(\widetilde{\zeta}) \end{pmatrix}\right\}$$

An asymptotic Evans function for high frequencies writes:

$$\lim_{|\zeta| \to \infty} \left|\frac{\mu_+^-(\widetilde{\zeta}) - \mu_-^+(\widetilde{\zeta})}{\Lambda(\widetilde{\zeta})}\right|.$$

Since there is $C > 0$ such that, for all $\rho \geq C > 0$, $\Re e\frac{\mu_+^-(\widetilde{\zeta})}{\Lambda(\widetilde{\zeta})} \geq C$ and $\Re e\frac{\mu_+^-(\widetilde{\zeta})}{\Lambda(\widetilde{\zeta})} \leq -C$, making $|\zeta| \to \infty$, we have:

$$\left|\frac{\mu_+^-(\widetilde{\zeta}) - \mu_-^+(\widetilde{\zeta})}{\Lambda(\widetilde{\zeta})}\right| \geq C' > 0.$$

Therefore, the Evans Condition is checked for high frequencies.

3.4.3 Computation of the asymptotic Evans function when $|\widetilde{\zeta}| \to 0^+$.

Remark that

$$\mu_+^-|_{\widetilde{\zeta}=0} = 0,$$

$$\mu_-^+|_{\widetilde{\zeta}=0} = 0.$$

As a result, the linear subspaces $\mathbb{E}_-(\mathbb{A}^+(\widetilde{\zeta}))$ and $\mathbb{E}_+(\mathbb{A}^-(\widetilde{\zeta}))$ cease to be well-defined. $\mathbb{A}^{\pm}(\widetilde{\zeta})$ appears in an ODE of the form:

$$\partial_z \begin{pmatrix} w^{\pm} \\ \partial_z w^{\pm} \end{pmatrix} = \mathbb{A}^{\pm}(\widetilde{\zeta}) \begin{pmatrix} w^{\pm} \\ \partial_z w^{\pm} \end{pmatrix} + F^{\pm},$$

We have then:

$$\partial_z \begin{pmatrix} w^{\pm} \\ \rho^{-1}\partial_z w^{\pm} \end{pmatrix} := \begin{pmatrix} 0 & \rho Id \\ \rho^{-1}(i\widetilde{\tau} + \widetilde{\gamma})Id & a^{\pm} \end{pmatrix} \begin{pmatrix} w^{\pm} \\ \rho^{-1}\partial_z w^{\pm} \end{pmatrix} := \rho\check{\mathbb{A}}(\check{\zeta}, \rho) \begin{pmatrix} w^{\pm} \\ \rho^{-1}\partial_z w^{\pm} \end{pmatrix},$$

where

$$\check{\mathbb{A}}^{\pm}(\check{\zeta}, \rho) := \begin{pmatrix} 0 & 1 \\ \rho^{-1}(i\check{\tau} + \check{\gamma}) & \rho^{-1}a^{\pm} \end{pmatrix}$$

with $\check{\tau} := \frac{\widetilde{\tau}}{\rho}$ and $\check{\gamma} := \frac{\widetilde{\gamma}}{\rho}$.
As reviewed earlier, a continuous extension of some positive and negative spaces of $\mathbb{A}^{\pm}$ has to be performed if we want to study the Evans function for low frequencies. These extended spaces are noted $\mathbb{E}_-^{lim}(\mathbb{A}^+)$ and $\mathbb{E}_+^{lim}(\mathbb{A}^-)$, and are computed as follows:

$$\mathbb{E}_-^{lim}(\mathbb{A}^+) = \mathbb{E}_-(\check{\mathbb{A}}^+)|_{\check{\tau}=1,\check{\gamma}=0,\rho=0},$$

and

$$\mathbb{E}_+^{lim}(\mathbb{A}^-) = \mathbb{E}_+(\check{\mathbb{A}}^-)|_{\check{\tau}=1,\check{\gamma}=0,\rho=0}.$$

The asymptotic Evans condition for low frequency writes then:

$$\mathbb{E}_-^{lim}(\mathbb{A}^+) \bigcap \mathbb{E}_+^{lim}(\mathbb{A}^-) = \{0\}.$$

Let us look at the negative eigenvalue of $\check{\mathbb{A}}^+(\check{\zeta}, \rho)$ that we will note $\check{\lambda}^+(\check{\zeta}, \rho)$ and compute its associated eigenvector:

$$\check{\mathbb{A}}^+ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \check{\lambda}^+ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

We get:

$$v_2 = \check{\lambda} v_1,$$

and multiplying by $\rho > 0$ the second coordinate of our vector gives:

$$(i\check{\tau} + \check{\gamma})v_1 + a^+ v_2 = \rho \check{\lambda} v_2$$

Making $\rho \to 0^+$, we obtain that:

$$\check{\lambda}^+(\check{\zeta}, \rho) = -\frac{i\check{\tau} + \check{\gamma}}{a^+}$$

As a result

$$\lim_{\rho \to 0^+} \mathbb{E}_- \left(\check{\mathbb{A}}^+(\check{\zeta}, \rho) \right) = Span \left\{ \begin{pmatrix} 1 \\ -\frac{i\check{\tau}+\check{\gamma}}{a^+} \end{pmatrix} \right\}$$

The same way, we have:

$$\lim_{\rho \to 0^+} \mathbb{E}_+ \left(\check{\mathcal{G}}^-(\check{\zeta}, \rho) \right) = Span \left\{ \begin{pmatrix} 1 \\ -\frac{i\check{\tau}+\check{\gamma}}{a^-} \end{pmatrix} \right\}$$

Taking $\check{\gamma} = 0$ and $\check{\tau} = 1$, since, by assumption, $a^- < 0$ and $a^+ > 0$ (otherwise the stability analysis for low frequencies would differ of the one we have just done), the Asymptotic Evans condition for low frequencies holds. This ends the proof of Proposition 3.3.4.

Chapter 4

Le Problème de Cauchy pour des Systèmes Hyperboliques Linéaires monodimensionnels avec une Discontinuité de coefficient pouvant présenter des modes expansifs : une approche visqueuse.

Ce chapitre reprend le papier [For07a] intitulé "The Cauchy Problem for 1-D Linear Nonconservative Hyperbolic Systems with possibly Expansive Discontinuity of the coefficient: a Viscous Approach" soumis à publication en septembre 2007.

Abstract

In this paper, we consider nonconservative Cauchy systems with discontinuous coefficients for a noncharacteristic boundary. The considered problems need not be the linearized of a shockwave on a shock front. We introduce then a viscous perturbation of the problem; the viscous solution u^ε depends of the small positive parameter ε. This problem, obtained by small viscous perturbation, is parabolic for fixed positive ε. Under some assumptions, incorporating a sharp spectral stability assumption, we prove the convergence, when $\varepsilon \to 0^+$, of u^ε towards the solution of a

well-posed hyperbolic limit problem. Our result is obtained, in the 1-D framework, for piecewise constant coefficients. Explicit examples of 2×2 systems satisfying our assumptions are given. They rely on a detailed analysis of our stability assumption (uniform Evans condition) for 2×2 systems.

The obtained result is new and generalizes the scalar expansive case solved in [For07d], where the considered hyperbolic operator was $\partial_t + a(x)\partial_x$, with $a(x) = a^+ > 0$ if $x > 0$ and $a(x) = a^- < 0$ if $x < 0$. A complete asymptotic description of the layer is given, at any order of approximation. In general, strong amplitude non-characteristic boundary layers form, which are localized on the area of discontinuity of the coefficient. Characteristic boundary layers, which appear along characteristic curves, also forms. Both type of boundary layers are polarized on specific disjoint linear subspaces.

4.1 Introduction.

Let us consider the 1-D linear hyperbolic system:

$$\begin{cases} \partial_t u + A(x)\partial_x u = f, & (t,x) \in \Omega, \\ u^\varepsilon|_{t=0} = h & . \end{cases}$$

where $\Omega = \{(t,x) \in (0,T) \times \mathbb{R}\}$, with $T > 0$ fixed once and for all. The unknown $u(t,x)$ belongs to $\mathbb{R}^N$ and A belongs to the set of $N \times N$ matrices with real coefficients $\mathcal{M}_N(\mathbb{R})$. A is assumed to satisfy:

$$A(x) = A^+\mathbf{1}_{x>0} + A^-\mathbf{1}_{x<0},$$

where A^+, A^-, are constant matrices in $\mathcal{M}_N(\mathbb{R})$. As we will detail later, since A is discontinuous through $\{x = 0\}$, this problem has no obvious sense. This problematic relates to many linear scalar works on analogous conservative problems. We can for instance refer to the works of Bouchut, James and Mancini in [BJ98], [BJM05]; by Poupaud and Rascle in [PR97] or by Diperna and Lions in [DL89]. We can also refer to [For07c] and [For07d] by Fornet. The common idea is that another notion of solution has to be introduced to deal with linear hyperbolic Cauchy problems with discontinuous coefficients. Note that almost all the papers cited before use a different approach to deal with the problem. Like in [For07c] and [For07d], we will opt for a small viscosity approach. Let us describe now the first result obtained in this paper. We consider the following viscous hyperbolic-parabolic problem:

$$\begin{cases} \partial_t u^\varepsilon + A(x)\partial_x u^\varepsilon - \varepsilon\partial_x^2 u^\varepsilon = f, & (t,x) \in \Omega, \\ u^\varepsilon|_{t=0} = h & , \end{cases} \tag{4.1.1}$$

where ε, commonly called viscosity, stands for a small positive parameter. Note well that, if we suppress the terms in $-\varepsilon\partial_x^2$ from our differential operator, the hyperbolic problem obtained has no obvious sense, because of the nonconservative product $A(x)\partial_x u$ not being well-defined when both u and A are discontinuous through $\{x = 0\}$.

The definition of such nonconservative product is of course crucial for defining a notion of weak solutions for such problems. It is an interesting question by itself, solved for instance in a quasi-linear framework by Dal Maso, LeFloch and Murat in [DMLM95] and by LeFloch and

Tzavaras in [LT99]. Existence and stability results in a neighboring framework of ours have been obtained by LeFloch ([LeF90]) in a 1-D scalar case and by Crasta and LeFloch ([CL02]) for 1-D systems. The equations studied in [LeF90] and [CL02] can be viewed as linear nonconservative problems with discontinuous coefficients; in these works the discontinuity of the coefficient is linked with a shockwave. Adopting a viscous approach allows us to avoid the difficult question of the definition of the nonconservative product in the linear framework.

In problem (4.1.1), the unknown is $u^\varepsilon(t,x) \in \mathbb{R}^N$, the source term f belongs to $H^\infty((0,T)\times\mathbb{R})$ and the Cauchy data h belongs to $H^\infty(\mathbb{R})$. We make the classical hyperbolicity assumption, plus we assume the boundary $\{x=0\}$ is noncharacteristic. In addition, we make a spectral stability assumption, which is an Uniform Evans Condition for a related problem. Last, we make an assumption ensuring that the limit hyperbolic problem satisfied by $u := \lim_{\varepsilon\to 0^+} u^\varepsilon$ is well-posed. The goal of Proposition 4.2.10 is to give, for $N=2$, examples of discontinuities of the coefficient (A^+, A^-) satisfying all our Assumptions. This Proposition relies on explicit algebraic computations of the Evans function performed in the case $N=2$.

Our assumptions do not forbid A^+ to have only positive eigenvalues and A^- of to have only negative eigenvalues. In this case, the discontinuity of the coefficient has a completely expansive setting. The question of the selection of a unique solution through a viscous approach was open, for this case, even for $N=1$, until [For07d]. Among other things, the result obtained previously in the scalar framework ([For07d]) is generalized to $N \in \mathbb{N}$ in this paper.

In order to describe our main result, let us introduce some notations. First, Σ is the linear subspace:

$$\Sigma := \left((A^+)^{-1} - (A^-)^{-1}\right)\left(\mathbb{E}_-(A^+)\bigcap \mathbb{E}_+(A^-)\right),$$

where, for instance,

$$\mathbb{E}_-(A^+) = \bigoplus_{\lambda_j^+<0} \ker\left(A^+ - \lambda_j^+ Id\right),$$

with λ_j^+ denoting the eigenvalues of A^+, which are real and semi-simple due to the hyperbolicity of the corresponding operator. $\mathbb{I}$ denotes the linear subspace given by:

$$\mathbb{I} := \mathbb{E}_-(A^-) \bigcap \mathbb{E}_+(A^+).$$

We choose, once for all, a linear subspace $\mathbb{V}$ such that:

$$\mathbb{E}_-(A^-) + \mathbb{E}_+(A^+) = \mathbb{I} \bigoplus \mathbb{V}.$$

We assume the following:

$$\mathbb{R}^N = \mathbb{I} \bigoplus \mathbb{V} \bigoplus \Sigma.$$

$\Pi_{\mathbb{I}}$ stands then for the linear projector on $\mathbb{I}$ parallel to $\mathbb{V} \bigoplus \Sigma$.

Note that, in [For07d], as a consequence of our assumptions, we had

$$\mathbb{R}^N = \mathbb{E}_-(A^-) \bigoplus \mathbb{E}_+(A^+) \bigoplus \Sigma,$$

which is the expression of our above assumption in the case $\mathbb{I} = \{0\}$ and also the expression of the uniform Lopatinski Condition in this special case.

This paper is mainly devoted to the proof of the following result: when $\varepsilon \to 0^+$, u^ε converges towards u in $L^2((0,T) \times \mathbb{R})$, where $u := u^+ \mathbf{1}_{x \geq 0} + u^- \mathbf{1}_{x<0}$ is the solution of the following **well-posed, even though not classical**, transmission problem:

$$\begin{cases} \partial_t u^- + A^- \partial_x u^- = f^-, & (t,x) \in (0,T) \times \mathbb{R}_-^*, \\ \partial_t u^+ + A^+ \partial_x u^+ = f^+, & (t,x) \in (0,T) \times \mathbb{R}_+^*, \\ u^+|_{x=0} - u^-|_{x=0} \in \Sigma, \\ \partial_x \Pi_{\mathbb{I}} u^+|_{x=0} - \partial_x \Pi_{\mathbb{I}} u^-|_{x=0} = 0, \\ u^-|_{t=0} = h^-, \\ u^+|_{t=0} = h^+. \end{cases} \tag{4.1.2}$$

$f^\pm$ and $h^\pm$ denotes respectively the restrictions of f and h to $\{\pm x > 0\}$

The proof of our convergence result splits into two parts. First, we construct an approximate solution of our viscous problem (4.1.1), then, we prove L^2 stability estimates via Kreiss-type Symmetrizers.

4.2 Nonconservative hyperbolic Cauchy problem with piecewise constant coefficients.

Let us recall the viscous parabolic problem (4.1.1):

$$\begin{cases} \partial_t u^\varepsilon + A(x)\partial_x u^\varepsilon - \varepsilon \partial_x^2 u^\varepsilon = f, & (t,x) \in \Omega, \\ u^\varepsilon|_{t=0} = h & . \end{cases}$$

We assume that $A(x) = A^+ \mathbf{1}_{x>0} + A^- \mathbf{1}_{x<0}$, with

Assumption 4.2.1. *[Hyperbolicity and Noncharacteristic boundary] A^+ and A^- are real diagonalizable constant matrices in $\mathcal{M}_N(\mathbb{R})$, $detA^- \neq 0$ and $detA^+ \neq 0$.*

Since the solution of the parabolic problem (4.1.1) is continuous, $\partial_x u^\varepsilon$ will not behave as a Dirac measure on $\{x = 0\}$. Moreover, since:

$$\varepsilon \partial_x^2 u^\varepsilon = f - \partial_t u^\varepsilon - A(x)\partial_x u^\varepsilon,$$

$\partial_x^2 u^\varepsilon$ got no Dirac measure on $\{x = 0\}$, thus implying the continuity of $\partial_x u^\varepsilon$ through $\{x = 0\}$. As a consequence, we get that u^ε is solution of (4.1.1) iff $(u_R^\varepsilon, u_L^\varepsilon)$ is solution of the following transmission problem:

$$\begin{cases} \partial_t u_R^\varepsilon + A^+\partial_x u_R^\varepsilon - \varepsilon\partial_x^2 u_R^\varepsilon = f_R, & \{x>0\}, t \in (0,T), \\ \partial_t u_L^\varepsilon + A^-\partial_x u_L^\varepsilon - \varepsilon\partial_x^2 u_L^\varepsilon = f_L, & \{x<0\}, t \in (0,T), \\ u_R^\varepsilon|_{x=0} - u_L^\varepsilon|_{x=0} = 0, & t \in (0,T), \\ \partial_x u_R^\varepsilon|_{x=0} - \partial_x u_L^\varepsilon|_{x=0} = 0, & t \in (0,T), \\ u_R^\varepsilon|_{t=0} = h_R(x), & \{x>0\}, \\ u_L^\varepsilon|_{t=0} = h_L(x), & \{x<0\} \quad . \end{cases} \tag{4.2.1}$$

The subscripts 'L' [resp 'R'] are used for the restrictions of the concerned functions to the **L**eft-hand side [resp **R**ight-hand side] of the boundary $\{x = 0\}$. We could refer to $\{x = 0\}$ as a boundary since the transmission problem (4.2.1) can be recast as the doubled problem on a half-space (4.2.2):

$$\begin{cases} \partial_t \widetilde{u}^\varepsilon + \widetilde{A}\partial_x \widetilde{u}^\varepsilon - \varepsilon\partial_x^2 \widetilde{u}^\varepsilon = \widetilde{f} & \{x>0\}, t \in (0,T) \\ \widetilde{\mathcal{M}}\widetilde{u}^\varepsilon|_{x=0} = 0 \\ \widetilde{u}^\varepsilon|_{t=0} = \widetilde{h} \end{cases} \tag{4.2.2}$$

where

$$\widetilde{u}^\varepsilon(t,x) = \begin{pmatrix} u_R^\varepsilon(t,x) \\ u_L^\varepsilon(t,-x) \end{pmatrix}$$

The new source term writes $\widetilde{f} = \begin{pmatrix} f_R(t,x) \\ f_L(t,-x) \end{pmatrix}$, and the new Cauchy data is $\widetilde{h} = \begin{pmatrix} h_R(t,x) \\ h_L(t,-x) \end{pmatrix}$, the new coefficient belongs to $\mathcal{M}_{2N}(\mathbb{R})$ and writes:

$$\widetilde{A} = \begin{pmatrix} A^+ & 0 \\ 0 & -A^- \end{pmatrix},$$

and the boundary operator writes

$$\widetilde{\mathcal{M}} = \begin{pmatrix} Id & -Id \\ \partial_x & \partial_x \end{pmatrix}.$$

Note that the classical parabolicity and hyperbolicity-parabolicity assumptions, see [Mét04] are trivially satisfied here.
Let $\mathbb{A}^\pm$ denote the matrices defined by:

$$\mathbb{A}^\pm = \begin{pmatrix} 0 & Id \\ (i\tau+\gamma)Id & A^\pm \end{pmatrix}.$$

We recall that we denote by $\mathbb{E}_+(\mathbb{A}^\pm)$ [resp $\mathbb{E}_-(\mathbb{A}^\pm)$] the linear subspace spanned by the generalized eigenvectors of $\mathbb{A}^\pm$ associated to the eigenvalues of $\mathbb{A}^\pm$ with positive [resp negative] real part and

$$det\left(\mathbb{E}_-(\mathbb{A}^+(\zeta)), \mathbb{E}_+(\mathbb{A}^-(\zeta))\right)$$

is the determinant obtained by taking orthonormal bases for both $\mathbb{E}_-(\mathbb{A}^+(\zeta))$ and $\mathbb{E}_+(\mathbb{A}^-(\zeta))$. We introduce the weight $\Lambda(\zeta)$ used to deal with high frequencies:

$$\Lambda(\zeta) = \left(1+\tau^2+\gamma^2\right)^{\frac{1}{2}}.$$

Let J_Λ be the mapping from $\mathbb{C}^N\times\mathbb{C}^N$ to $\mathbb{C}^N\times\mathbb{C}^N$ $(u,v)\mapsto(u,\Lambda^{-1}v)$. We can introduce now the scaled negative and positive spaces of matrices $\mathbb{A}^\pm$:

$$\widetilde{\mathbb{E}}_\pm(\mathbb{A}^\pm) := J_\Lambda \mathbb{E}_\pm(\mathbb{A}^\pm).$$

Our stability assumption writes:

Assumption 4.2.2 (Uniform Evans Condition)**.**
$(\widetilde{\mathcal{H}}^\varepsilon, \widetilde{\mathcal{M}})$ *satisfies the Uniform Evans Condition which means that, for all* $\zeta = (\tau, \gamma) \in \mathbb{R} \times \mathbb{R}^+ - \{0_{\mathbb{R}^2}\}$, *there holds:*

$$\left| det\left(\widetilde{\mathbb{E}}_-(\mathbb{A}^+(\zeta)), \widetilde{\mathbb{E}}_+(\mathbb{A}^-(\zeta))\right)\right| \geq C > 0.$$

In a different framework than ours, the study of such stability assumption has been done in many papers. For example, we can refer the reader to the paper of Gardner and Zumbrun ([GZ98]), Guès, Métivier, Williams and Zumbrun ([GMWZ05]), Métivier and Zumbrun ([MZ05]), Rousset ([Rou03]) and finaly Serre ([Ser05]) . A more recent reference is [BGSZ06] by Benzoni-Gavage, Serre and Zumbrun.

Assumption 4.2.3. *There holds:*

$$\left(\mathbb{E}_-(A^-) + \mathbb{E}_+(A^+)\right) \bigoplus \Sigma = \mathbb{R}^N.$$

Keeping in mind that the linear subspace $\mathbb{I}$ is defined by $\mathbb{I} := \mathbb{E}_-(A^-) \bigcap \mathbb{E}_+(A^+)$, Assumption 4.2.3 also writes:

$$\mathbb{R}^N = \mathbb{I} \bigoplus \mathbb{V} \bigoplus \Sigma. \tag{4.2.3}$$

We introduce then the projectors associated to this decomposition, that we respectively note: $\Pi_{\mathbb{I}}$, $\Pi_{\mathbb{V}}$ and Π_Σ.

After introducing the necessary notations, we will formulate an assumption concerning the structure of the discontinuity (A^-, A^+).

By assumption 4.2.1, there are two nonsingular matrices P^+, P^- and two diagonal matrices D^+ and D^- such that $D^+ = (P^+)^{-1}A^+P^+$ and $D^- = (P^-)^{-1}A^-P^-$. We denote then $\mathbb{J} := \mathbb{E}_-(D^-) \bigcap \mathbb{E}_+(D^+)$. Let us choose two linear subspaces of $\mathbb{R}^N$, $\mathbb{V}_1$ and $\mathbb{V}_2$ such that:

$$\mathbb{V}_1 \bigoplus \mathbb{J} = \mathbb{E}_+(D^+),$$

and

$$\mathbb{V}_2 \bigoplus \mathbb{J} = \mathbb{E}_-(D^-).$$

Assumption 4.2.4 (Structure of discontinuity)**.**
There holds:

$$P^+\mathbb{V}_1 \bigoplus \left(P^+\mathbb{J} + P^-\mathbb{J}\right) \bigoplus P^-\mathbb{V}_2 \bigoplus \Sigma = \mathbb{R}^N$$

Moreover, the mapping

$$M := \begin{pmatrix} \Pi_{\mathbb{I}} P^+(D^+)^{-1} & -\Pi_{\mathbb{I}} P^-(D^-)^{-1} \\ P^+ & -P^- \end{pmatrix}$$

from $\mathbb{J}\times\mathbb{J}$ *into* $\mathbb{I}\times(P^+\mathbb{J}+P^-\mathbb{J})$ *defines an isomorphism between* $\mathbb{J}\times\mathbb{J}$ *and* $\mathbb{I}\times(P^+\mathbb{J}+P^-\mathbb{J})$ *. Finally, we assume that:* $\dim \mathbb{E}_-(A^+)\bigcap \mathbb{E}_+(A^-) = \dim\Sigma$.

Remark 4.2.5. *If* $\dim\mathbb{I} = \dim\mathbb{J}$, *then Assumption 4.2.4 implies that* $P^+\mathbb{J} = P^-\mathbb{J}$.

Let us make a remark concerning 2×2 strictly hyperbolic systems. We take $A^- = \begin{pmatrix} d_1^- & 0 \\ 0 & d_2^- \end{pmatrix}$ and $A^+ = \begin{pmatrix} d_1^+ & \alpha \\ 0 & d_2^+ \end{pmatrix}$, with $d_1^- < 0$ and $d_1^+ > 0$ and $\alpha \in \mathbb{R}^*$. We have $P^- = Id$, $P^+ = \begin{pmatrix} 1 & 1 \\ 0 & \frac{d_1^+ - d_2^+}{-\alpha} \end{pmatrix}$, $D^- = A^-$ and $D^+ = \begin{pmatrix} d_1^+ & 0 \\ 0 & d_2^+ \end{pmatrix}$. As a consequence, $\mathbb{J} = Span\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Moreover $\mathbb{V}_2 = \{0\}$ because $\mathbb{J} = \mathbb{E}_-(A^-)$. Since $\mathbb{E}_+(D^+) = \mathbb{R}^2$, we take $\mathbb{V}_1 = Span\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Moreover, $\mathbb{E}_-(A^+)\bigcap\mathbb{E}_+(A^-) = \{0\}$ thus $\Sigma = \{0\}$. We check then easily that, like before, if we take $d_2^- > 0$ and $d_2^+ < 0$, Assumption 4.2.4 is not satisfied for any $\alpha \neq 0$. More general examples of this form will be analyzed thanks to a new assumption about the structure of the discontinuity, that will be introduced now.

The general assumption is Assumption 4.2.4. However, we also state a special set of sufficient conditions, which are easier to check in some cases. They write:

Assumption 4.2.6 (Structure of discontinuity, sufficient version)**.**
We assume that:

- $\dim\Sigma = \dim\mathbb{E}_-(A^+)\bigcap\mathbb{E}_+(A^-)$.

- $A^-\mathbb{I} = \mathbb{I}$
- $A^+\mathbb{I} = \mathbb{I}$
- $\ker((A^+)^{-1} - (A^-)^{-1}) \bigcap \mathbb{I} = \{0\}$
- $\mathbb{E}_-((Id - \Pi_{\mathbb{I}})A^-(Id - \Pi_{\mathbb{I}})) \bigoplus \mathbb{E}_+((Id - \Pi_{\mathbb{I}})A^+(Id - \Pi_{\mathbb{I}})) \bigoplus \Sigma = \mathbb{V} \bigoplus \Sigma$
- $\dim \mathbb{E}_-(A^+) \bigcap \mathbb{E}_+(A^-) = \dim \Sigma$.

Assumption 4.2.6 is a sufficient condition for Assumption 4.2.4 to hold. While this assumption is less general than Assumption 4.2.4, it is in general easier to check.

If A^- has only negative eigenvalues and A^+ has only positive eigenvalues (totally expansive case), this assumption reduces to:

$$\ker\left((A^+)^{-1} - (A^-)^{-1}\right) \bigcap \mathbb{I} = \{0\}.$$

Since $\mathbb{I} = \mathbb{R}^N$ in the totally expansive case, the assumption also writes:

$$det\left((A^+)^{-1} - (A^-)^{-1}\right) \neq 0.$$

Moreover, if both A^+ and A^- are diagonal or if we make the same assumptions as in [For07d], this assumption trivially holds.
Let us now give an example for which Assumption 4.2.4 holds for strictly hyperbolic 2×2 systems. Let us take $A^- = \begin{pmatrix} d_1^- & 0 \\ 0 & d_2^- \end{pmatrix}$ and $A^+ = \begin{pmatrix} d_1^+ & \alpha \\ 0 & d_2^+ \end{pmatrix}$, with $d_1^- < 0$, $d_2^- > 0$, $d_1^+ > 0$, $d_2^+ > 0$ and $\alpha \in \mathbb{R}^*$. We assume moreover that the eigenvalues of A^- and A^+ are all distinct. Note well that there is no lack of generality in considering A^- diagonal since, by change of basis, we can diagonalize either A^- or A^+. We have then $\mathbb{E}_-(A^-) = Span\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbb{E}_+(A^+) = \mathbb{R}^2$, which implies that: $\mathbb{I} = Span\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We have moreover $A^+\mathbb{I} = A^-\mathbb{I} = \mathbb{I}$. Since $\mathbb{E}_-(A^+) = \{0\}$ and $\mathbb{E}_+(A^-) = Span\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we get that $\Sigma = \{0\}$.

Moreover, $((A^+)^{-1} - (A^-)^{-1}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{d_1^+} - \frac{1}{d_1^-} \\ 0 \end{pmatrix}$, which implies that:

$$Ker((A^+)^{-1} - (A^-)^{-1}) \bigcap \mathbb{I} = \{0\}.$$

Let us take $\mathbb{V} = \mathbb{I}^\perp = Span \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. There holds: $\mathbb{I} \bigoplus \mathbb{V} = \mathbb{R}^2$. We can make this choice whenever $\Sigma = \{0\}$. We have now to check that:

$$\mathbb{E}_-(\Pi_\mathbb{V} A^- \Pi_\mathbb{V}) \bigoplus \mathbb{E}_+(\Pi_\mathbb{V} A^+ \Pi_\mathbb{V}) = \mathbb{V}.$$

Let us take $v \in \mathbb{V}$, we have then $v = \Pi_\mathbb{V} v$. $\Pi_\mathbb{V}$ writes:

$$\Pi_\mathbb{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Actually $\Pi_\mathbb{V} A^- \Pi_\mathbb{V} = \begin{pmatrix} 0 & 0 \\ 0 & d_2^- \end{pmatrix}$ and $\Pi_\mathbb{V} A^+ \Pi_\mathbb{V} = \begin{pmatrix} 0 & 0 \\ 0 & d_2^+ \end{pmatrix}$ thus $\mathbb{E}_-(\Pi_\mathbb{V} A^- \Pi_\mathbb{V}) = \{0\}$ and $\mathbb{E}_+(\Pi_\mathbb{V} A^+ \Pi_\mathbb{V}) = \mathbb{V}$, hence we have checked that Assumption 4.2.4 holds for the considered matrices A^- and A^+.
Let us discuss this example further. Firstly, this example works more generally for $sign(d_2^-) = sign(d_2^+)$. Secondly, if we took $d_2^- > 0$ and $d_2^+ < 0$ Assumption 4.2.4 is not satisfied for any $\alpha \neq 0$, but is satisfied for $\alpha = 0$ independently of the signs of $d_1^\pm$ and $d_2^\pm$. Finally, Assumption 4.2.4 is satisfied in the completely outgoing case i.e if we take $d_2^- < 0$ and $d_2^+ > 0$.

Remark 4.2.7. *The uniform Evans condition is a criterion of stability that seems difficult to check. This stability assumption has been studied in several papers as it is central, among other things, in the study of the stability of shockwaves. As mentioned in [GZ98], a sufficient condition for the Evans condition to hold begins difficult to establish for systems with $N \geq 3$. However, for large systems, computational methods have been proposed for this purpose, see [HSZ06] for a recent approach.*

We will now state some of our results concerning the study of the Evans Condition. For $N = 2$, we will give very simple sufficient conditions for Evans-stability and Evans-instability.
Without lack of generality, we can assume that A^- is diagonal. We

denote then by $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ the normalized eigenvectors of A^+. Let us define $q := \dim \Sigma$.

Proposition 4.2.8. *For $N = 2$, i.e for 2×2 systems, and whether $q = 0$, $q = 1$, or $q = 2$, the problem associated to the choice of matrices (A^+, A^-) satisfying: $sign(ad) = -sign(bc)$ or $ad = 0$ or $bc = 0$ is Evans-Stable (but not necessarily uniformly Evans-stable).*

In the following Proposition, $\lambda_1^\pm$ and $\lambda_2^\pm$ denote the two eigenvalues of $A^\pm$.

Proposition 4.2.9. *Provided that the matrices (A^+, A^-) are such that: $a, b, c, d > 0$, $bc > ad$ and $\lambda_1^+ = -\lambda_2^+ < 0$, $\lambda_1^- = -\lambda_2^- < 0$; the associated problem is strongly Evans-unstable, in the sense that the Evans function vanishes for some (τ, γ) with $\tau \in \mathbb{R}$ and $\gamma > 0$.*

As a consequence of the stability analysis performed in section 4.3, there holds:

Proposition 4.2.10. *Let P denote a nonsingular matrix in $\mathcal{M}_2(\mathbb{R})$, then the matrices A^- and A^+ defined by:*

$$A^- = P^{-1} \begin{pmatrix} d_1^- & 0 \\ 0 & d_2^- \end{pmatrix} P$$

and

$$A^+ = P^{-1} \begin{pmatrix} d_1^+ & \alpha \\ 0 & d_2^+ \end{pmatrix} P$$

with $d_1^- < 0$, $d_1^+ > 0$ and $\alpha \in \mathbb{R} - \{0\}$ satisfy all our assumptions iff either d_2^+ and d_2^- have the same sign or $d_2^- < 0$ and $d_2^+ > 0$.

4.2.1 Construction of an approximate solution as a BKW expansion.

We will construct an approximate solution of problem (4.2.1) at any order. This construction will show that, if $\mathbb{E}_-(A^-) \bigcap \mathbb{E}_+(A^+) \neq \{0\}$, weak amplitude characteristic boundary layers forms similarly to [For07c]. Moreover, if $\mathbb{E}_-(A^+) \bigcap \mathbb{E}_+(A^-) \neq \{0\}$, large noncharacteristic boundary layers forms on the area of discontinuity of the coefficients: $\{x = 0\}$.

Let us note $\Omega_L = \{(t,x) \in (0,T) \times \mathbb{R}^{*-}\}$ and $\Omega_R = \{(t,x) \in (0,T) \times \mathbb{R}^{*+}\}$. $u^\varepsilon_{app,L}$ [resp $u^\varepsilon_{app,R}$] denotes the restriction of the solution to Ω_L [resp Ω_R]. We will construct $u^\varepsilon_{app,L} \in C^1(\Omega_L) \bigcap L^2(\Omega_L)$ and $u^\varepsilon_{app,R} \in C^1(\Omega_R) \bigcap L^2(\Omega_R)$. To that aim, let us first introduce some notations. The matrix A^- [resp A^+] has N_- [resp N_+] negative [resp positive] eigenvalues. Let $\mu_1^-, \dots, \mu_{N_-}^-$ be the negative eigenvalues of A^- sorted by increasing order and $\mu_1^+, \dots, \mu_{N_+}^+$ be the positive eigenvalues of A^+ sorted by decreasing order. We introduce the following partition of Ω_L :

$$\Omega_L = \mathcal{C}_L \bigsqcup \left(\bigsqcup_{j=0}^{N_-} \Omega_L^j \right),$$

where

$$\mathcal{C}_L := \bigcup_{j=1}^{N_-} \left\{ (t,x) \in \Omega_L : x - \mu_j^- t = 0 \right\},$$

$$\Omega_L^0 := \left\{ (t,x) \in \Omega_L : x - \mu_1^- t < 0 \right\},$$

and for all $1 \leq j \leq N_- - 1$

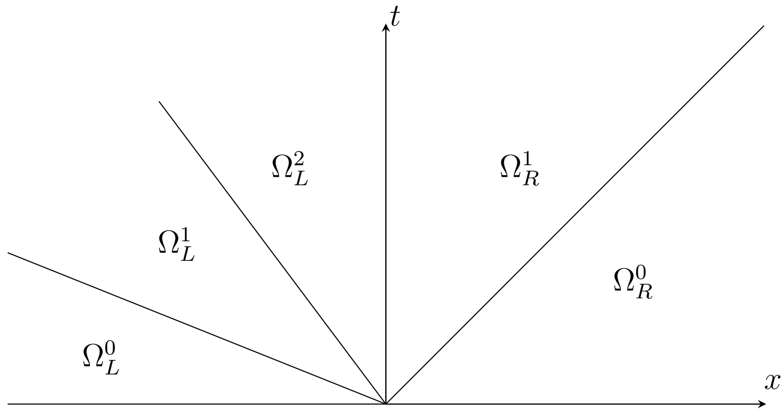
$$\Omega_L^j := \left\{ (t,x) \in \Omega_L : \mu_j^- t < x < \mu_{j+1}^- t < 0 \right\},$$

and

$$\Omega_L^{N_-} := \left\{ (t,x) \in \Omega_L : x - \mu_{N_-}^- t > 0 \right\}.$$

On the right hand side, we do the analogous partition:

$$\Omega_R = \mathcal{C}_R \bigsqcup \left(\bigsqcup_{j=0}^{N_-} \Omega_R^j \right).$$



This drawing shows the case where $N_- = 2$ and $N_+ = 1$.

Remark 4.2.11. *Note that the boundary layer profiles serve the purpose of correcting singularities possibly forming in the small viscosity limit on $\{x = 0\}$, $\mathcal{C}_R$, and $\mathcal{C}_L$. We will give an ansatz incorporating such terms. We do not assume that any compatibility condition is checked, thus, generally speaking, each line composing $\mathcal{C}_R$ and $\mathcal{C}_L$ supports singularities of $u := \lim_{\varepsilon\to 0^+} u^\varepsilon$. A natural question is to localize the singularities linked with the expansive modes of the discontinuity. If $e_j \in \mathbb{V}_2$, (e_j is the j^{th} vector of the canonical basis of $\mathbb{R}^N$), then the possible singularities of u on $\{(t,x) \in \Omega_L : x - \lambda_j^- t = 0\}$, where λ_j^- stands for the j^{th} diagonal coefficient of D^-, are not induced by any expansive mode. The same way, if $e_j \in \mathbb{V}_1$, then the possible singularities of u on $\{(t,x) \in \Omega_R : x - \lambda_j^+ t = 0\}$, where λ_j^+ stands for the j^{th} diagonal coefficient of D^+, are not induced by any expansive mode.*

Let us introduce the different profiles and their ansatz. We will construct separately the restriction $u^{\varepsilon,j}_{app,L}$ of $u^\varepsilon_{app,L}$ to each Ω_L^j for $0 \leq j \leq N_-$ so that, the different pieces of approximate solution glued back together gives the approximate solution $u^\varepsilon_{app,L} \in C^1(\Omega_L) \bigcap L^2(\Omega_L)$.

$$u^{\varepsilon,j}_{app,L}(t,x) = \sum_{n=0}^{M} \left(\underline{\mathbf{U}}^j_{n,L}(t,x) + \mathbf{U}^{*,j}_{n,L}\left(t, \frac{x}{\varepsilon}\right)\right) \sqrt{\varepsilon}^n$$
$$+ \mathbf{U}^{c,j}_{n,L}\left(t, \frac{x - \mu_1^- t}{\sqrt{\varepsilon}}, \ldots, \frac{x - \mu_{N_-}^- t}{\sqrt{\varepsilon}}\right) \sqrt{\varepsilon}^n$$

Actually, depending on the value of j, the ansatz can be written in a simplified manner, but we rather give here a generic ansatz valid for all

j. Somewhat related ansatzs can be found in [For07c] and [For07d]. The $\underline{\mathbf{U}}^j_{n,L}$ belongs to $H^\infty\left(\Omega^j_L\right)$. Given that $\mathbf{U}^{*,j}_{n,L}=0$ except for $j=N_-$, we will denote $\mathbf{U}^{*,N_-}_{n,L}$ by $\mathbf{U}^*_{n,L}$. The noncharacteristic boundary layer profiles $\mathbf{U}^*_{n,L,+}(t,z)$ belongs to $e^{\delta z}H^\infty((0,T)\times\mathbb{R}^*_-)$, for some $\delta>0$. Let us review the characteristic boundary layer profiles $\mathbf{U}^{c,j}_{n,L,+}\left(t,\theta^1_L,\dots,\theta^{N_-}_L\right)$. For $j=0$, we can use the simplified ansatz $\mathbf{U}^{c,0}_{n,L,+}\left(t,\theta^1_L\right)$ with $\mathbf{U}^{c,0}_{n,L,+}$ belonging to $e^{\delta\theta^1_L}H^\infty((0,T)\times\mathbb{R}^*_-)$, for some $\delta>0$. For $j=N_-$ we can adopt the simplified ansatz $\mathbf{U}^{c,N_-}_{n,L,+}\left(t,\theta^{N_-}_L\right)$ with $\mathbf{U}^{c,N_-}_{n,L,+}$ belonging to $e^{-\delta\theta^{N_-}_L}H^\infty((0,T)\times\mathbb{R}^*_+)$, for some $\delta>0$. For $1\le j\le N_--1$, we have also the simplified ansatz: $\mathbf{U}^{c,j}_{n,L,+}\left(t,\theta^j_L,\theta^{j+1}_L\right)$. Let us denote by $E_{\mu^-_j}$ the eigenspace of A^- associated to the eigenvalue μ^-_j. We have then the following decomposition of $\mathbb{R}^N$:

$$\mathbb{R}^N=\bigoplus_{j=1}^{N_-}E_{\mu^-_j}\bigoplus\mathbb{E}_+(A^-),$$

we have thus the associated equality on the projectors:

$$Id=\sum_{j=1}^{N_-}\Pi^-_j+\Pi_{\mathbb{E}_+(A^-)}.$$

(4.2.4) $$\mathbf{U}^{c,j}_{n,L,+}\left(t,\theta^j_L,\theta^{j+1}_L\right)=\Pi^-_j\mathbf{U}^{c,j}_{n,L,+}\left(t,\theta^j_L\right)+\Pi^-_{j+1}\mathbf{U}^{c,j+1}_{n,L,+}\left(t,\theta^{j+1}_L\right).$$

where $\Pi^-_j\mathbf{U}^{c,j}$ belongs to to $e^{-\delta\theta^j_L}H^\infty((0,T)\times\mathbb{R}^*_+)$, for some $\delta>0$, $\Pi^-_{j+1}\mathbf{U}^{c,j+1}$ belongs to to $e^{\delta\theta^{j+1}_L}H^\infty((0,T)\times\mathbb{R}^*_-)$, for some $\delta>0$. This means that on each subset, after projection, the involved layer profile depends only of one fast characteristic dependent variable.
In a similar way, we have:

$$u^{\varepsilon,j}_{app,R}(t,x)=\sum_{n=0}^{M}\left(\underline{\mathbf{U}}^j_{n,R}(t,x)+\mathbf{U}^{*,j}_{n,R}\left(t,\frac{x}{\varepsilon}\right)\right)\sqrt{\varepsilon}^n$$

$$+\mathbf{U}^{c,j}_{n,R}\left(t,\frac{x-\mu^+_1t}{\sqrt{\varepsilon}},\dots,\frac{x-\mu^+_{N_+}t}{\sqrt{\varepsilon}}\right)\sqrt{\varepsilon}^n$$

with an ansatz identical to the one exposed before.

Let us explain the different steps of the construction of the approximate solution. We begin by constructing the profiles $(\mathbf{U}_j^*, \underline{\mathbf{U}}_j)$ in cascade, the characteristic profiles $\underline{\mathbf{U}}_j^c$ are then computed as a last step.

Plugging the approximate solution into the equation an identifying the terms with the same power in ε, we obtain our profiles equations. $\left(\mathbf{U}_{R,0}^*, \mathbf{U}_{L,0}^*\right)$ is solution of the following ODE in z:

$$\begin{cases} A^+\partial_z \mathbf{U}_{R,0}^* - \partial_z^2 \mathbf{U}_{R,0}^* = 0, & \{z > 0\}, \\ A^-\partial_z \mathbf{U}_{L,0}^* - \partial_z^2 \mathbf{U}_{L,0}^* = 0, & \{z < 0\}, \\ \mathbf{U}_{R,0}^*|_{z=0} - \mathbf{U}_{L,0}^*|_{z=0} = -(\underline{\mathbf{U}}_{R,0}|_{x=0} - \underline{\mathbf{U}}_{L,0}|_{x=0}), \\ \partial_z \mathbf{U}_{R,0}^*|_{z=0} - \partial_z \mathbf{U}_{L,0}^*|_{z=0} = 0. \end{cases}$$

Since we search for $\mathbf{U}_{R,0}^*$ and $\mathbf{U}_{L,0}^*$ tending towards zero when $z \to \pm\infty$, it is equivalent to solve:

$$\begin{cases} \partial_z \mathbf{U}_{R,0}^* - A^+ \mathbf{U}_{R,0}^* = 0, & \{z > 0\}, \\ \partial_z \mathbf{U}_{L,0}^* - A^- \mathbf{U}_{L,0}^* = 0, & \{z < 0\}, \\ \mathbf{U}_{R,0}^*|_{z=0} - \mathbf{U}_{L,0}^*|_{z=0} = -(\underline{\mathbf{U}}_{R,0}|_{x=0} - \underline{\mathbf{U}}_{L,0}|_{x=0}), \\ \partial_z \mathbf{U}_{R,0}^*|_{z=0} - \partial_z \mathbf{U}_{L,0}^*|_{z=0} = 0. \end{cases}$$

Applying $\Pi_{\mathbb{I}}$ to our equations on $\mathbf{U}_{R,0}^*$ and $\mathbf{U}_{L,0}^*$, we get that:
$\Pi_{\mathbb{I}} \mathbf{U}_{R,0}^* = e^{A^+ z} \Pi_{\mathbb{I}} \mathbf{U}_{R,0}^*|_{z=0}$, with

$$\Pi_{\mathbb{I}} \mathbf{U}_{R,0}^*|_{z=0} \in \mathbb{E}_-(A^+) \bigcap \mathbb{E}_-(A^-) \bigcap \mathbb{E}_+(A^+) = \{0\},$$

and $\mathbf{U}_{L,0}^* = e^{A^- z} \Pi_{\mathbb{I}} \mathbf{U}_{L,0}^*|_{z=0}$, with

$$\Pi_{\mathbb{I}} \mathbf{U}_{L,0}^*|_{z=0} \in \mathbb{E}_+(A^-) \bigcap \mathbb{E}_-(A^-) \bigcap \mathbb{E}_+(A^+) = \{0\}.$$

We obtain then that $\Pi_{\mathbb{I}} \mathbf{U}_{L,0}^* = \Pi_{\mathbb{I}} \mathbf{U}_{R,0}^* = 0$. The same argument apply at any order, giving that, for all $0 \leq j \leq M$, there holds:

$$\Pi_{\mathbb{I}} \mathbf{U}_{L,j}^* = \Pi_{\mathbb{I}} \mathbf{U}_{R,j}^* = 0.$$

We have just proved that $\mathbf{U}_{R,0}^* = (\Pi_{\mathbb{V}} + \Pi_{\Sigma}) \mathbf{U}_{R,0}^*$ and that $\mathbf{U}_{L,0}^* = (\Pi_{\mathbb{V}} + \Pi_{\Sigma}) \mathbf{U}_{L,0}^*$. Moreover $\mathbf{U}_{R,0}^* = e^{A^+ z} \mathbf{U}_{R,0}^*|_{z=0}$, with $\mathbf{U}_{R,0}^*|_{z=0} \in \mathbb{E}_-(A^+)$

and $\mathbf{U}^*_{L,0} = e^{A^- z}\mathbf{U}^*_{L,0}|_{z=0}$, with $\mathbf{U}^*_{L,0}|_{z=0} \in \mathbb{E}_+(A^-)$. From the second boundary condition, by using the equation, we get that:

$$A^+\mathbf{U}^*_{R,0}|_{z=0} = A^-\mathbf{U}^*_{L,0}|_{z=0} \in \mathbb{E}_-(A^+)\bigcap \mathbb{E}_+(A^-),$$

let us denote by σ_0' this quantity. Returning to the first boundary condition, this leads to:

$$\underline{\mathbf{U}}_{R,0}|_{x=0} - \underline{\mathbf{U}}_{L,0}|_{x=0} = -\left((A^+)^{-1} - (A^-)^{-1}\right)\sigma_0' := \sigma_0,$$

with $\sigma_0' \in \mathbb{E}_-(A^+)\bigcap \mathbb{E}_+(A^-)$, which gives:

$$\underline{\mathbf{U}}_{R,0}|_{x=0} - \underline{\mathbf{U}}_{L,0}|_{x=0} \in \Sigma.$$

For fixed $\sigma_0 \in \Sigma$, the equations giving the profiles $\mathbf{U}^*_{L,0}$ and $\mathbf{U}^*_{R,0}$ are well-posed since we have assumed that $\dim\Sigma = \dim \mathbb{E}_-(A^+)\bigcap \mathbb{E}_+(A^-)$, which is equivalent to $\ker\left((A^+)^{-1} - (A^-)^{-1}\right)\bigcap\left(\mathbb{E}_-(A^+)\bigcap \mathbb{E}_+(A^-)\right) = \{0\}$.

We shall now introduce the solution $(\underline{\mathbf{U}}_{L,0}, \underline{\mathbf{U}}_{R,0})$ of the following hyperbolic problem, which is also the limiting hyperbolic problem as ε goes to zero:

$$\begin{cases} \partial_t\underline{\mathbf{U}}_{L,0} + A^-\partial_x\underline{\mathbf{U}}_{L,0} = f^L, & (t,x)\in\Omega_L, \\ \partial_t\underline{\mathbf{U}}_{R,0} + A^+\partial_x\underline{\mathbf{U}}_{R,0} = f^R, & (t,x)\in\Omega_R, \\ \underline{\mathbf{U}}_{R,0}|_{x=0} - \underline{\mathbf{U}}_{L,0}|_{x=0} \in \Sigma, \\ \partial_x\Pi_{\mathbb{I}}\underline{\mathbf{U}}_{R,0}|_{x=0} - \partial_x\Pi_{\mathbb{I}}\underline{\mathbf{U}}_{L,0}|_{x=0} = 0, \\ \underline{\mathbf{U}}_{L,0}|_{t=0} = h^L, \\ \underline{\mathbf{U}}_{R,0}|_{t=0} = h^R. \end{cases} \tag{4.2.5}$$

Under our assumptions, this problem is well-posed, as we will prove now. The profiles $\underline{\mathbf{U}}^j_{L,0}$ for $0 \le j \le N_-$ are the restriction of $\underline{\mathbf{U}}_{L,0}$ to Ω^j_L. The same way, the profiles $\underline{\mathbf{U}}^j_{R,0}$ for $0 \le j \le N_+$ are the restriction of $\underline{\mathbf{U}}_{R,0}$ to Ω^j_R.

Proposition 4.2.12. *If Assumption 4.2.4 is checked, which means there holds*

$$P^+\mathbb{V}_1 \bigoplus \left(P^+\mathbb{J} + P^-\mathbb{J}\right) \bigoplus P^-\mathbb{V}_2 \bigoplus \Sigma = \mathbb{R}^N,$$

$$\dim \mathbb{E}_-(A^+)\bigcap \mathbb{E}_+(A^-) = \dim \Sigma,$$

and the mapping

$$M := \begin{pmatrix} \Pi_{\mathbb{I}} P^+(D^+)^{-1} & -\Pi_{\mathbb{I}} P^-(D^-)^{-1} \\ P^+ & -P^- \end{pmatrix}$$

from $\mathbb{J}\times\mathbb{J}$ *into* $\mathbb{I}\times(P^+\mathbb{J}+P^-\mathbb{J})$ *defines an isomorphism between* $\mathbb{J}\times\mathbb{J}$ *and* $\mathbb{I}\times(P^+\mathbb{J}+P^-\mathbb{J})$, *then the transmission problem* (4.2.5) *has a unique solution.*

Proof. For the sake of simplicity let us denote $u_L := \underline{\mathbf{U}}_{L,0}$ and $u_R := \underline{\mathbf{U}}_{R,0}$. Given our assumptions, there are two nonsingular matrices P^+, P^- and two diagonal matrices D^+ and D^- such that $D^+ = (P^+)^{-1}A^+P^+$ and
$D^- = (P^-)^{-1}A^-P^-$. Taking $v_R := (P^+)^{-1}u_R$ and $v_L := (P^-)^{-1}u_L$, we obtain that (v_L, v_R) is solution the equivalent transmission problem:

$$\begin{cases} \partial_t v_R + D^+\partial_x v_R = (P^+)^{-1} f_R, & \{x>0\}, \\ \partial_t v_L + D^-\partial_x v_L = (P^-)^{-1} f_L, & \{x<0\}, \\ P^+ v_R|_{x=0} - P^- v_L|_{x=0} \in \Sigma, \\ \partial_x \Pi_{\mathbb{I}} P^+ v_R|_{x=0} - \partial_x \Pi_{\mathbb{I}} P^- v_L|_{x=0} = 0, \\ v_L|_{t=0} = (P^-)^{-1} h_L, \\ v_R|_{t=0} = (P^+)^{-1} h_R. \end{cases}$$

Let us denote by $\Pi_{\mathbb{E}_-(D^+)}$ and $\Pi_{\mathbb{E}_+(D^+)}$ the projector associated to the decomposition:

$$\mathbb{R}^N = \mathbb{E}_-(D^+)\bigoplus \mathbb{E}_+(D^+),$$

we define likewise $\Pi_{\mathbb{E}_-(D^-)}$ and $\Pi_{\mathbb{E}_+(D^-)}$. We recall that we have as well the decomposition (4.2.3). Equation

$$\partial_t v_R + D^+\partial_x v_R = (P^+)^{-1} f_R, \qquad \{x>0\},$$

splits into:

$$v_R = \Pi_{\mathbb{E}_+(D^+)} v_R + \Pi_{\mathbb{E}_-(D^+)} v_R,$$

$$\partial_t(\Pi_{\mathbb{E}_+(D^+)} v_R) + D^+\partial_x(\Pi_{\mathbb{E}_+(D^+)} v_R) = \Pi_{\mathbb{E}_+(D^+)}(P^+)^{-1} f_R, \qquad \{x>0\},$$

and

$$\partial_t(\Pi_{\mathbb{E}_-(D^+)} v_R) + D^+\partial_x(\Pi_{\mathbb{E}_-(D^+)} v_R) = \Pi_{\mathbb{E}_-(D^+)}(P^+)^{-1} f_R, \qquad \{x>0\}.$$

These problems being diagonal, they are tantamount to N scalar, easily solved, independent equations; which shows that: $\Pi_{\mathbb{E}_-(D^+)}v_R$ and $\Pi_{\mathbb{E}_+(D^-)}v_L$ are directly computed from the equation without boundary conditions. Contrary to them, $\Pi_{\mathbb{E}_-(D^+)}v_R$ and $\Pi_{\mathbb{E}_+(D^-)}v_L$ can be computed only when the traces $\Pi_{\mathbb{E}_-(D^+)}v_R|_{x=0}$ and $\Pi_{\mathbb{E}_+(D^-)}v_L|_{x=0}$ are known. The well-posedness of our problem reduces to the algebraic well-posedness of a linear system whose equations are our boundary conditions and the unknowns are the traces $\Pi_{\mathbb{E}_-(D^+)}v_R|_{x=0}$ and $\Pi_{\mathbb{E}_+(D^-)}v_L|_{x=0}$. The boundary condition states that there is $\sigma \in \Sigma$ such that:

$$P^+\Pi_{\mathbb{E}_+(D^+)}v_R - P^-\Pi_{\mathbb{E}_-(D^-)}v_L + \sigma = -P^+\Pi_{\mathbb{E}_-(D^+)}v_R + P^-\Pi_{\mathbb{E}_+(D^-)}v_L.$$

Let us recall a piece of Assumption 4.2.4:

$$P^+\mathbb{V}_1 \bigoplus \left(P^+\mathbb{J} + P^-\mathbb{J}\right) \bigoplus P^-\mathbb{V}_2 \bigoplus \Sigma = \mathbb{R}^N. \tag{4.2.6}$$

By (4.2.6) and since P^+ and P^- are nonsingular, we get the value of the traces on the boundary of:

$$\Pi_1\Pi_{\mathbb{E}_+(D^+)}v_R,$$

$$\Pi_2\Pi_{\mathbb{E}_-(D^-)}v_L,$$

and

$$P^+\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}v_R - P^-\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}v_L,$$

as well as the value of σ. To compute the traces $u_R|_{x=0}$ and $u_L|_{x=0}$, we only lack the knowledge of $\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}v_R|_{x=0}$ and $\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}v_L|_{x=0}$. By the equation, there holds:

$$\partial_x v_R = (D^+)^{-1}\left((P^+)^{-1}f_R - \partial_t v_R\right),$$

$$\partial_x v_L = (D^-)^{-1}\left((P^-)^{-1}f_L - \partial_t v_L\right).$$

The boundary condition $\Pi_{\mathbb{I}}\partial_x v_R|_{x=0} - \Pi_{\mathbb{I}}\partial_x v_L|_{x=0} = 0$ gives then a relation of the form:

$$\Pi_{\mathbb{I}}P^+(D^+)^{-1}\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}\partial_t v_R|_{x=0} - \Pi_{\mathbb{I}}P^-(D^-)^{-1}\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}\partial_t v_L|_{x=0} = q$$

where q is a known continuous function of $t \in (0,T)$, with values polarized on the linear subspace $\mathbb{I}$. Since we have as well

$$P^+\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_+(D^+)}\partial_t v_R|_{x=0} - P^-\Pi_{\mathbb{J}}\Pi_{\mathbb{E}_-(D^-)}\partial_t v_L|_{x=0} = q'$$

where q' is a known continuous function of $t \in (0, T)$. By Assumption 4.2.4, for all fixed t there is only one $\partial_t \Pi_{\mathbb{J}} \Pi_{\mathbb{E}_+(D^+)} v_R|_{x=0}(t)$ and $\partial_t \Pi_{\mathbb{J}} \Pi_{\mathbb{E}_-(D^-)} v_L|_{x=0}(t)$ solution of this linear system of two equations with two unknowns. Moreover, q and q' depending continuously of $t \in (0, T)$, it is also the case for $\partial_t \Pi_{\mathbb{J}} \Pi_{\mathbb{E}_+(D^+)} v_R|_{x=0}$ and $\partial_t \Pi_{\mathbb{J}} \Pi_{\mathbb{E}_-(D^-)} v_L|_{x=0}$. We have thus:

$$\Pi_{\mathbb{J}} \Pi_{\mathbb{E}_+(D^+)} v_R|_{x=0} = \Pi_{\mathbb{J}} \Pi_{\mathbb{E}_+(D^+)} h(0) + \int_0^t \partial_t \Pi_{\mathbb{J}} \Pi_{\mathbb{E}_+(D^+)} v_R|_{x=0}(s)\ ds,$$

and

$$\Pi_{\mathbb{J}} \Pi_{\mathbb{E}_-(D^-)} v_L|_{x=0} = \Pi_{\mathbb{J}} \Pi_{\mathbb{E}_-(D^-)} h(0) + \int_0^t \partial_t \Pi_{\mathbb{J}} \Pi_{\mathbb{E}_-(D^-)} v_L|_{x=0}(s)\ ds,$$

which achieves the computation of the traces $g_L := v_L|_{x=0}$ and $g_R := v_R|_{x=0}$. We obtain then that the hyperbolic problem (4.2.5), which satisfies nonclassical transmission conditions on the boundary, is actually equivalent to solve two classical well-posed mixed hyperbolic problem with Dirichlet boundary conditions. $u_R = P^+ v_R$, where v_R is solution of:

$$\begin{cases} \partial_t v_R + D^+ \partial_x v_R = (P^+)^{-1} f_R, & \{x > 0\}, \\ v_R|_{x=0} = g_R, \\ v_R|_{t=0} = (P^+)^{-1} h_R. \end{cases}$$

This problem is well-posed because $\Pi_{\mathbb{E}_-(D^+)} g_R$ is incidentally the trace $\Pi_{\mathbb{E}_-(D^+)} v_R|_{x=0}$ computed from the equation without boundary condition. As a consequence, this problem also rewrites:

$$\begin{cases} \partial_t v_R + D^+ \partial_x v_R = (P^+)^{-1} f_R, & \{x > 0\}, \\ \Pi_{\mathbb{E}_+(D^+)} v_R|_{x=0} = \Pi_{\mathbb{E}_+(D^+)} g_R. \\ v_R|_{t=0} = (P^+)^{-1} h_R. \end{cases}$$

which is a mixed hyperbolic problem satisfying a Uniform Lopatinski condition. The same way v_L is the solution of the following mixed hyperbolic problem satisfying a Uniform Lopatinski condition:

$$\begin{cases} \partial_t v_L + D^- \partial_x v_L = (P^-)^{-1} f_L, & \{x < 0\}, \\ \Pi_{\mathbb{E}_-(D^-)} v_L|_{x=0} = \Pi_{\mathbb{E}_-(D^-)} g_L. \\ v_L|_{t=0} = (P^-)^{-1} h_L, \end{cases}$$

and u_L is obtained by: $u_L = P^+ v_L$, which shows that problem (4.2.5) is well-posed.

□

Proof of the well-posedness of the transmission problem (4.2.5) **under Assumption 4.2.6**

There holds:

(4.2.7)

$$\begin{cases} \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0} + \Pi_{\mathbb{I}} A^- \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0} = \Pi_{\mathbb{I}} f^L - \Pi_{\mathbb{I}} A^- \partial_x (\Pi_{\mathbb{V}} + \Pi_{\Sigma}) \underline{\mathbf{U}}_{L,0}, \quad \{x < 0\}. \\ \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0} + \Pi_{\mathbb{I}} A^+ \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0} = \Pi_{\mathbb{I}} f^R - \Pi_{\mathbb{I}} A^+ \partial_x (\Pi_{\mathbb{V}} + \Pi_{\Sigma}) \underline{\mathbf{U}}_{R,0}, \quad \{x > 0\}. \\ \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} - \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0} = 0, \\ \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} - \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0} = 0, \\ \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{t=0} = \Pi_{\mathbb{I}} h^L, \\ \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{t=0} = \Pi_{\mathbb{I}} h^R. \end{cases}$$

Hence, by Assumption 4.2.6, we have:

(4.2.8)

$$\begin{cases} \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0} + A^- \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0} = \Pi_{\mathbb{I}} f^L - \Pi_{\mathbb{I}} A^- \partial_x (\Pi_{\mathbb{V}} + \Pi_{\Sigma}) \underline{\mathbf{U}}_{L,0}, \quad \{x < 0\}. \\ \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0} + A^+ \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0} = \Pi_{\mathbb{I}} f^R - \Pi_{\mathbb{I}} A^+ \partial_x (\Pi_{\mathbb{V}} + \Pi_{\Sigma}) \underline{\mathbf{U}}_{R,0}, \quad \{x > 0\}. \\ \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} - \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0} = 0, \\ \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} - \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0} = 0, \\ \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{t=0} = \Pi_{\mathbb{I}} h^L, \\ \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{t=0} = \Pi_{\mathbb{I}} h^R. \end{cases}$$

Let us now introduce $\underline{\mathbf{V}}_{L,0} = (Id - \Pi_{\mathbb{I}}) \underline{\mathbf{U}}_{L,0}$, $\underline{\mathbf{V}}_{R,0} = (Id - \Pi_{\mathbb{I}}) \underline{\mathbf{U}}_{R,0}$, applying then $(Id - \Pi_{\mathbb{I}})$ to our equation, we get the following:

$$\begin{cases} \partial_t \underline{\mathbf{V}}_{L,0} + (Id - \Pi_{\mathbb{I}}) M^- \partial_x \underline{\mathbf{V}}_{L,0} = (Id - \Pi_{\mathbb{I}}) f^L, \qquad \{x < 0\}. \\ \partial_t \underline{\mathbf{V}}_{R,0} + (Id - \Pi_{\mathbb{I}}) M^+ \partial_x \underline{\mathbf{V}}_{R,0} = (Id - \Pi_{\mathbb{I}}) f^R, \qquad \{x > 0\}. \\ \underline{\mathbf{V}}_{R,0}|_{x=0} - \underline{\mathbf{V}}_{L,0}|_{x=0} \in \Sigma, \\ \underline{\mathbf{V}}_{L,0}|_{t=0} = (Id - \Pi_{\mathbb{I}}) h^L, \\ \underline{\mathbf{V}}_{R,0}|_{t=0} = (Id - \Pi_{\mathbb{I}}) h^R. \end{cases}$$

Referring the reader to the analysis performed in the multi-D case treated in [For07d] for further details, this mixed hyperbolic problem is well-posed provided that it satisfies the Uniform Lopatinski Condition stating that

$$\mathbb{E}_-((Id - \Pi_{\mathbb{I}})M^-) \bigoplus \mathbb{E}_+((Id - \Pi_{\mathbb{I}})M^+) \bigoplus \Sigma = \mathbb{V} \bigoplus \Sigma.$$

As we will see, we can now compute the solution of (4.2.8). Indeed there is an unique

$$\mathbf{g}(t) := \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} = \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0},$$

which depends continuously of $t \in (0,T)$, satisfying our boundary conditions provided that

$$\mathrm{Ker}((A^+)^{-1} - (A^-)^{-1}) \bigcap \mathbb{I} = \{0\}.$$

Indeed, by using the equation, we get that $\partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} - \partial_x \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{L,0}|_{x=0} = 0$ writes as well:

$$\left((A^+)^{-1} - (A^-)^{-1}\right) \partial_t \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0} = q'',$$

where q'' stands for a known function continuous in t. As a result, we obtain that:

$$\Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0}(t) = \Pi_{\mathbb{I}} \underline{\mathbf{U}}_{R,0}|_{x=0}(t) = \Pi_{\mathbb{I}} h(0) + \int_0^t \mathbf{g}(s)\ ds,$$

which proves the well-posedness of the hyperbolic problem (4.2.5) under Assumption 4.2.6.

Since Assumption 4.2.4 being checked is a sufficient but also necessary condition in order for problem (4.2.5) to be well-posed, we get then that:

$$[\text{Assumption } 4.2.6 \Rightarrow \text{Assumption } 4.2.4].$$

Since the problem (4.2.5) is well-posed, $u_L|_{x=0} - u_R|_{x=0} := \sigma_0 \in \Sigma$ is known and thus $\mathbf{U}^*_{L,0}$ and $\mathbf{U}^*_{R,0}$ as well. This scheme of construction can be carried out at any order. Let us show how the the other profiles are constructed:

$$\begin{cases} A^+ \partial_z \mathbf{U}^*_{R,1} - \partial_z^2 \mathbf{U}^*_{R,1} = 0, & \{z > 0\}, \\ A^- \partial_z \mathbf{U}^*_{L,1} - \partial_z^2 \mathbf{U}^*_{L,1} = 0, & \{z < 0\}, \\ \mathbf{U}^*_{R,1}|_{z=0} - \mathbf{U}^*_{L,1}|_{x=0} = -(\underline{\mathbf{U}}_{R,1}|_{x=0} - \underline{\mathbf{U}}_{L,1}|_{x=0}), \\ \partial_z \mathbf{U}^*_{R,1}|_{z=0} - \partial_z \mathbf{U}^*_{L,1}|_{z=0} = 0. \end{cases}$$

$$\begin{cases} A^+\partial_z \mathbf{U}^*_{R,2} - \partial_z^2 \mathbf{U}^*_{R,2} = -\partial_t \mathbf{U}^*_{R,0}, & \{z>0\}, \\ A^-\partial_z \mathbf{U}^*_{L,2} - \partial_z^2 \mathbf{U}^*_{L,2} = -\partial_t \mathbf{U}^*_{L,0}, & \{z<0\}, \\ \mathbf{U}^*_{R,2}|_{z=0} - \mathbf{U}^*_{L,2}|_{x=0} = -(\underline{\mathbf{U}}_{R,2}|_{x=0} - \underline{\mathbf{U}}_{L,2}|_{x=0}), \\ \partial_z \mathbf{U}^*_{R,2}|_{z=0} - \partial_z \mathbf{U}^*_{L,2}|_{z=0} = -\left(\partial_x \underline{\mathbf{U}}_{R,0}|_{x=0} - \partial_x \underline{\mathbf{U}}_{L,0}|_{x=0}\right). \end{cases}$$

$\Pi_2 \mathbf{U}^*_{L,2} = \Pi_2 \mathbf{U}^*_{R,2} = 0$, which does not contradict our previous computations since $\mathbf{\Pi}_2 \left(\partial_x \underline{\mathbf{U}}_{R,0}|_{x=0} - \partial_x \underline{\mathbf{U}}_{L,0}|_{x=0}\right) = 0$. Actually for $n \geq 2$, we have:

$$\begin{cases} A^+\partial_z \mathbf{U}^*_{R,n} - \partial_z^2 \mathbf{U}^*_{R,n} = -\partial_t \mathbf{U}^*_{R,n-2}, & \{z>0\}, \\ A^-\partial_z \mathbf{U}^*_{L,n} - \partial_z^2 \mathbf{U}^*_{L,n} = -\partial_t \mathbf{U}^*_{L,n-2}, & \{z<0\}, \\ \mathbf{U}^*_{R,n}|_{z=0} - \mathbf{U}^*_{L,n}|_{x=0} = -(\underline{\mathbf{U}}_{R,n}|_{x=0} - \underline{\mathbf{U}}_{L,n}|_{x=0}), \\ \partial_z \mathbf{U}^*_{R,n}|_{z=0} - \partial_z \mathbf{U}^*_{L,n}|_{z=0} = -\left(\partial_x \underline{\mathbf{U}}_{R,n-2}|_{x=0} - \partial_x \underline{\mathbf{U}}_{L,n-2}|_{x=0}\right). \end{cases}$$

$(\underline{\mathbf{U}}_{L,n}, \underline{\mathbf{U}}_{R,n})$ are given by:

$$\begin{cases} \partial_t \underline{\mathbf{U}}_{L,n} + A^-\partial_x \underline{\mathbf{U}}_{L,n} = \partial_x^2 \underline{\mathbf{U}}_{L,n-2}, & \{x<0\}. \\ \partial_t \underline{\mathbf{U}}_{R,n} + A^+\partial_x \underline{\mathbf{U}}_{R,n} = \partial_x^2 \underline{\mathbf{U}}_{R,n-2}, & \{x>0\}. \\ \underline{\mathbf{U}}_{R,n}|_{x=0} - \underline{\mathbf{U}}_{L,n}|_{x=0} \in p_n + \Sigma, \\ \partial_x \Pi_2 \underline{\mathbf{U}}_{R,n}|_{x=0} - \partial_x \Pi_2 \underline{\mathbf{U}}_{L,n}|_{x=0} = 0, \\ \underline{\mathbf{U}}_{L,n}|_{t=0} = 0, \\ \underline{\mathbf{U}}_{R,n}|_{t=0} = 0. \end{cases} \tag{4.2.9}$$

where p_n is computed using the equations on $\mathbf{U}^*_{R,n}$ and $\mathbf{U}^*_{L,n}$. This mixed hyperbolic problem is well-posed for the same reasons as the mixed hyperbolic problems giving $(\underline{\mathbf{U}}_{L,0}, \underline{\mathbf{U}}_{R,0})$. The profiles $\underline{\mathbf{U}}^j_{L,n}$ for $0 \leq j \leq N_-$ are the restriction of $\underline{\mathbf{U}}_{L,n}$ to Ω^j_L. The same way, the profiles $\underline{\mathbf{U}}^j_{R,n}$ for $0 \leq j \leq N_+$ are the restriction of $\underline{\mathbf{U}}_{R,n}$ to Ω^j_R.

Referring to (4.2.4), we have actually to compute the profiles $\Pi_j^- \mathbf{U}^{c,j}_{n,L,\pm}(t, \theta^j_L)$ and $\Pi_j^+ \mathbf{U}^{c,j}_{n,R,\pm}(t, \theta^j_R)$. Since the profiles equations satisfied by $\Pi_j^- \mathbf{U}^{c,j}_{n,L,\pm}$ and $\Pi_j^+ \mathbf{U}^{c,j}_{n,R,\pm}$ are of the same form, we will only focus on the computation of the profiles $\mathbf{U}^{c,\pm}_{L,n}(t, z_j) := \Pi_j^- \mathbf{U}^{c,j}_{n,L,\pm}(t, \theta^j_L)$ for some j. Observe that, the pieces of solutions $(\underline{\mathbf{U}}_{L,j}, \underline{\mathbf{U}}_{R,j})$ glued together compose in general a function belonging to $C^0((0,T)\times\mathbb{R})$ but not to $C^1((0,T)\times\mathbb{R})$. Since the characteristic profiles allow the glued together approximate solution to belong to $C^1((0,T)\times\mathbb{R})$, computing the characteristics layer profiles amounts to solve equations of the form:

$$\begin{cases} \partial_t \mathbf{U}^{c,+}_{L,n} - \partial^2_{z_j}\mathbf{U}^{c,+}_{L,n} = 0, & \{z_j > 0\}, \\ \partial_t \mathbf{U}^{c,-}_{L,n} - \partial^2_{z_j}\mathbf{U}^{c,-}_{L,n} = 0, & \{z_j < 0\}, \\ [\mathbf{U}^c_{L,n}]_j(t) = -[\underline{\mathbf{U}}_{L,n}]_{\Gamma_j}(t), \quad \forall t\in(0,T), \\ [\partial_x \mathbf{U}^c_{L,n}]_j(t) = -\dfrac{1}{2}\left([\partial_x \underline{\mathbf{U}}_{L,n-1}]_{\Gamma_j}(t) + [\partial_x U^c_{L,n-1}]_j(t)\right), \quad \forall t\in(0,T), \\ \mathbf{U}^{c+}_{L,n}|_{t=0} = 0, \\ \mathbf{U}^{c-}_{L,n}|_{t=0} = 0, \end{cases}$$

where $[\omega]_j(t) = \lim_{z_j\to 0^+}\omega(t,z_j) - \lim_{z_j\to 0^-}\omega(t,z_j)$ and $[\omega']_{\Gamma_j}(t) = \lim_{x\to\mu_j^- t, x>\mu_j^- t}\omega'(t,x) - \lim_{x\to\mu_j^- t, x<\mu_j^- t}\omega'(t,x)$. These profiles equations are clearly well-posed, using the same argument used in [For07d]. To sum up, we have constructed
$u^\varepsilon_{app} := u^\varepsilon_{R,app}\mathbf{1}_{x\geq 0} + u^\varepsilon_{L,app}\mathbf{1}_{x<0}$ such that:

$$\begin{cases} \partial_t u^\varepsilon_{app} + A(x)\partial_x u^\varepsilon_{app} - \varepsilon\partial^2_x u^\varepsilon_{app} = f + \varepsilon^M R^\varepsilon, \quad (t,x)\in\Omega, \\ u^\varepsilon_{app}|_{t=0} = h. \end{cases}$$

4.2.2 Stability estimates.

This time, we will rather note

$$u^\varepsilon_{app} := u^{\varepsilon+}_{app}(t,x)\mathbf{1}_{x>0} + u^{\varepsilon-}_{app}(t,-x)\mathbf{1}_{x<0}.$$

By linearity, the error equation writes, for $w^\varepsilon = u^\varepsilon_{app} - u^\varepsilon$:

$$\begin{cases} \partial_t w^\varepsilon + A(x)\partial_x w^\varepsilon - \varepsilon\partial^2_x w^\varepsilon = \varepsilon^M R^\varepsilon, \qquad (t,x)\in\Omega, \\ w^\varepsilon|_{t=0} = 0. \end{cases}$$

Since our method of estimation comes from pseudodifferential calculus, we have to perform a tangential Fourier-Laplace transform of the problem. For this purpose, it is necessary to extend the definition of

our error, in order for it to be defined for all time $t \in \mathbb{R}$. We first perform an extension of w^ε to $\{t < 0\}$ as follows: $\widetilde{w}^\varepsilon := \begin{cases} w^\varepsilon \ on \ (0,T) \\ 0 \ on \ t<0 \end{cases}$
but, for fixed positive ε,
$w^\varepsilon \in C((0,T) : L^2(\mathbb{R}))$ and $w^\varepsilon|_{t=0} = 0$ thus $\widetilde{w}^\varepsilon$ belongs to
$C((-\infty, T] : L^2(\mathbb{R}))$. Moreover, $\partial_t \widetilde{w}^\varepsilon$ has no Dirac measure on $\{t = 0\}$ and thus $\widetilde{w}^\varepsilon$ is solution of:

$$\partial_t \widetilde{w}^\varepsilon + A(x)\partial_x \widetilde{w}^\varepsilon - \varepsilon \partial_x^2 \widetilde{w}^\varepsilon = \varepsilon^M \widetilde{R}^\varepsilon, \qquad (t,x) \in (-\infty, T] \times \mathbb{R},.$$

where $\widetilde{R}^\varepsilon := \begin{cases} R^\varepsilon & if & t \in (0,T), \\ 0 & on & t < 0. \end{cases}$
Finally, we denote by $\underline{\widetilde{R}}^\varepsilon$, $\widetilde{R}^\varepsilon$ extended by 0 outside $(0,T) \times \mathbb{R}$. Let us now proceed with the extension of our error to $t > T$. We call by $\underline{\widetilde{w}}^\varepsilon$ the unique solution of:

$$\begin{cases} \mathcal{H}\underline{\widetilde{w}}^\varepsilon - \varepsilon \partial_x^2 \underline{\widetilde{w}}^\varepsilon = \varepsilon^M \underline{\widetilde{R}}^\varepsilon, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ \underline{\widetilde{w}}^\varepsilon|_{t<0} = 0. \end{cases} \tag{4.2.10}$$

Note well that the restriction of $\underline{\widetilde{w}}^\varepsilon$ to Ω is w^ε. For the sake of simplicity, we will still denote $\underline{\widetilde{w}}^\varepsilon$ [resp $\underline{\widetilde{R}}^\varepsilon$] by w^ε [resp R^ε] in what follows.

To begin with, let us rewrite the problem (4.2.10) in a convenient form. w^ε is solution of:

$$\partial_t w^\varepsilon + A(x)\partial_x w^\varepsilon - \varepsilon \partial_x^2 w^\varepsilon = \varepsilon^M R^\varepsilon, \qquad (t,x) \in \mathbb{R} \times \mathbb{R},$$

We denote then by $\hat{w}^{\varepsilon\pm} := \mathcal{F}(e^{-\gamma t} w^{\varepsilon\pm})$ and $\hat{R}^{\varepsilon\pm} := \mathcal{F}(e^{-\gamma t} R^{\varepsilon\pm})$, where $\mathcal{F}$ stands for the tangential Fourier transform (with respect to t) and the $\pm$ superscripts indicates restrictions to $\{\pm x > 0\}$, we have then:

$$\begin{cases} (i\tau + \gamma)\hat{w}^{\varepsilon+} + A^+ \partial_x \hat{w}^{\varepsilon+} - \varepsilon \partial_x^2 \hat{w}^{\varepsilon+} = \varepsilon^M \hat{R}^{\varepsilon+}, & \{x > 0\}, \\ (i\tau + \gamma)\hat{w}^{\varepsilon-} + A^- \partial_x \hat{w}^{\varepsilon-} - \varepsilon \partial_x^2 \hat{w}^{\varepsilon-} = \varepsilon^M \hat{R}^{\varepsilon-}, & \{x < 0\}, \\ \hat{w}^{\varepsilon+}|_{x=0} - \hat{w}^{\varepsilon-}|_{x=0} = 0, \\ \partial_x \hat{w}^{\varepsilon+}|_{x=0} - \partial_x \hat{w}^{\varepsilon-}|_{x=0} = 0. \end{cases} \tag{4.2.11}$$

Remark that, by taking γ big enough, the restrictions of the solution w^ε of (4.2.10) to $\{\pm x > 0\}$ are given by:

$$w^{\varepsilon\pm} = e^{\gamma t} \mathcal{F}^{-1}(\hat{w}^{\varepsilon\pm}),$$

where $(\hat{w}^{\varepsilon+}, \hat{w}^{\varepsilon-})$ are the solutions of the transmission problem (4.2.11).

Taking $W^{\varepsilon\pm}(i\tau+\gamma, x) = \begin{pmatrix} \hat{w}^{\varepsilon\pm} \\ \varepsilon\partial_x \hat{w}^{\varepsilon\pm} \end{pmatrix}$,

$$\begin{cases} \partial_x W^{\varepsilon+} = \begin{pmatrix} \partial_x \hat{w}^{\varepsilon+} \\ \varepsilon\partial_x^2 \hat{w}^{\varepsilon+} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon} Id \\ (i\tau+\gamma) & \frac{1}{\varepsilon} A^+ \end{pmatrix} \begin{pmatrix} \hat{w}^{\varepsilon+} \\ \varepsilon\partial_x \hat{w}^{\varepsilon+} \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon^M \hat{R}^{\varepsilon+} \end{pmatrix}, \\ \partial_x W^{\varepsilon-} = \begin{pmatrix} \partial_x \hat{w}^{\varepsilon-} \\ \varepsilon\partial_x^2 \hat{w}^{\varepsilon-} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon} Id \\ (i\tau+\gamma) & \frac{1}{\varepsilon} A^- \end{pmatrix} \begin{pmatrix} \hat{w}^{\varepsilon-} \\ \varepsilon\partial_x \hat{w}^{\varepsilon-} \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon^M \hat{R}^{\varepsilon-} \end{pmatrix}, \\ W^{\varepsilon+}|_{x=0} - W^{\varepsilon-}|_{x=0} = 0. \end{cases}$$

We note $\zeta = (\tau, \gamma)$ and $\widetilde{\zeta} = (\varepsilon\tau, \varepsilon\gamma)$. Multiplying the previous equation by ε gives:

$$\begin{cases} \partial_z W^{\varepsilon+} - \mathbb{A}^+(\widetilde{\zeta}) W^{\varepsilon+} = G^+, \quad \{z>0\}, \\ \partial_z W^{\varepsilon-} - \mathbb{A}^-(\widetilde{\zeta}) W^{\varepsilon-} = \widetilde{G}^-, \quad \{z<0\}, \\ W^{\varepsilon+}|_{z=0} = W^{\varepsilon-}|_{z=0}, \end{cases}$$

where

$$G^\pm = \begin{pmatrix} 0 \\ \varepsilon^{M+1} \hat{R}^{\varepsilon\pm} \end{pmatrix},$$

and z stands for the fast variable $\frac{x}{\varepsilon}$. From this point onwards, since nothing differs from the proof of stability by symmetrizers done in [For07d], we give the result:

Proposition 4.2.13. *There is $C>0$ such that for all $0<\varepsilon<1$, there holds:*

$$\|u^\varepsilon - u^\varepsilon_{app}\|_{L^2(\Omega)} \le C\varepsilon^{M-1}.$$

4.2.3 The main result.

We recall that u^ε stands for the solution of the viscous problem (4.1.1) and $u := u^+ \mathbf{1}_{x\ge 0} + u^- \mathbf{1}_{x<0}$, where (u^+, u^-) is solution of the well-posed transmission problem (4.1.2) or (4.2.5).

Theorem 4.2.14. *u^ε converges towards u in $L^2(\Omega)$ as ε tends to zero. More precisely, there is $C > 0$, independent of ε such that:*

$$\|u^\varepsilon - u\|_{L^2((0,T)\times\mathbb{R})} \leq C\varepsilon.$$

Proof. By construction of our approximate solution u^ε_{app}, we have:

$$\|u^\varepsilon - u\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon).$$

Hence, by constructing our approximate solution at a sufficient order M, Proposition 4.2.13 ends the proof. □

4.3 Stability study for 2×2 nonconservative systems.

In this chapter, our goal is to analyze the uniform Evans condition for 2×2 systems. We limit ourselves to this framework due to the fast increasing complexity of the computations with the size of the systems. This analysis is not trivial to perform, as witness, even for 2×2 systems, a sufficient and necessary reformulation of the Evans Condition, not involving any frequencies, has yet to be found out. Our point here is to give a brief overview of the link existing between the matrices A^- and A^+ and the uniform Evans condition being checked. As a result of our study, the uniform Evans Condition does not appear as a very restrictive assumption, but, on the other hand, is not always satisfied. The uniform Evans Condition writes as the nonvanishing of an Evans function for a given range of frequencies. This Evans function is a determinant that can be written in several equivalent ways. D and $\widetilde{D}$ are two equivalent Evans functions iff, for all $\zeta \neq 0$,

$$D(\zeta) = 0 \Leftrightarrow \widetilde{D}(\zeta) = 0.$$

We will begin by giving the expression of an Evans function for medium frequencies, then we will introduce asymptotic Evans functions for $|\zeta| \to \infty$ (high frequencies) and $|\zeta| \to 0^+$ (low frequencies). Our results for 2×2 systems are divided the same way. The study of the low frequency behavior is the more technical, since some arguments break down due to eigenvalues crossing the imaginary axis. The specific analysis for low frequencies involves the continuous extension of some linear subspaces intervening in the formulation of the Evans function. A part of our analysis is devoted to the computation of these extensions for some 2×2 systems. During our study, we achieve the proof of Proposition (4.2.10).

4.3.1 Spectral analysis of the symbol $\mathbb{A}^{\pm}$.

The expression of an Evans function relies on the computation of the linear subspaces $\mathbb{E}_{-}(\mathbb{A}^{+})$ and $\mathbb{E}_{+}(\mathbb{A}^{-})$. An important point is that, except for low frequencies, the eigenvalues of $\mathbb{A}^{+}$ and $\mathbb{A}^{-}$ do not cross the imaginary axis. $\mathbb{A}^{+}$ and $\mathbb{A}^{-}$ have both N eigenvalues with positive real part and N eigenvalues with negative real part. As a consequence, if the Evans condition holds, for all ζ in an open subset not containing $\{0\}$, there holds: $\mathbb{E}_{-}(\mathbb{A}^{+}) \bigoplus \mathbb{E}_{+}(\mathbb{A}^{-}) = \mathbb{C}^{2N}$. We will now show that the eigenvectors of $\mathbb{A}^{\pm}$ can be deduced from the eigenvectors of $A^{\pm}$. Denote by v_i^{+} [resp v_i^{-}]the normalized eigenvector associated to the eigenvalue λ_i^{+} of A^{+} [resp λ_i^{-} of A^{-}]. The eigenvectors of $\mathbb{A}^{+}$ associated to the eigenvalues with negative real parts, denoted by $(\mu_i^{+})_{1\leq i\leq N}$, are given by:

$$\left(\mathbf{w}_i^{+}\right)_{1\leq i\leq N} := \begin{pmatrix} v_i^{+} \\ \mu_i^{+} v_i^{+} \end{pmatrix}_{1\leq i\leq N}.$$

Likewise, the eigenvectors of $\mathbb{A}^{+}$ associated to the eigenvalues with positive real parts, noted $(\mu_i^{+})_{N+1\leq i\leq 2N}$, are given by:

$$\left(\mathbf{w}_i^{+}\right)_{N+1\leq i\leq 2N} := \begin{pmatrix} v_i^{+} \\ \mu_{N+i}^{+} v_i^{+} \end{pmatrix}_{N+1\leq i\leq 2N}.$$

The family $\left(\mathbf{w}_i^{+}\right)_{1\leq i\leq N}$ is a basis of $\mathbb{E}_{-}(\mathbb{A}^{+})$. Moreover, μ_i^{+} satisfy:

$$\mu_i^{+2} - \lambda_j^{+}\mu_i^{+} - (i\tau + \gamma) = 0.$$

Proof. Denote μ an eigenvalue of $\mathbb{A}^{+}$ and $\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$ an eigenvector associated to μ.

$$\begin{cases} \mathbf{v}_2 = \mu \mathbf{v}_1 \\ A^{+}\mathbf{v}_1 = \dfrac{\mu^2 - (i\tau + \gamma)}{\mu}\mathbf{v}_1 \end{cases}$$

Since $\mathbf{v}_1 = 0_{\mathbb{R}^N} \Rightarrow \mathbf{v} = 0_{\mathbb{C}^{2N}}$, $\mathbf{v}_1$ is an eigenvector of A^{+} associated to the eigenvalue $\frac{\mu^2-(i\tau+\gamma)}{\mu}$. Hence there is $1 \leq j \leq N$ such that $\lambda_j^{+} =$

$\frac{\mu^2-(i\tau+\gamma)}{\mu}$. We will show here that, for all $(\tau,\gamma)\neq 0$, the eigenvalues of $\mathbb{A}^+$ are all semi-simple and that N of them have positive real part and N of them have negative real part. This result is deduced from the fact that we can associate to each eigenvalues of A^+ two eigenvalues of $\mathbb{A}^+$: one with positive real part and one with negative real part. Moreover, for each eigenvalue of $\mathbb{A}^+$ the associated eigenvector can be directly constructed by using the eigenvector associated to the corresponding eigenvalue of A^+ as stated above. The eigenvalues of $\mathbb{A}^+$ are the roots of P defined by:

$$P(\mu)=\mu^2-\lambda\mu-(i\tau+\gamma).$$

Note that the roots of P^+ are:

$$\mu_- = \frac{1}{2}\left(\lambda - sign\left(cos(\theta^+/2)\right)\sqrt{r^+}e^{i(\theta^+/2)}\right),$$

$$\mu_+ = \frac{1}{2}\left(\lambda + sign\left(cos(\theta^+/2)\right)\sqrt{r^+}e^{i(\theta^+/2)}\right).$$

where $r^+=\sqrt{(\lambda^2+4\gamma)^2+16\tau^2}$ and $\theta^+=\arctan\frac{4\tau}{\lambda^2+4\gamma}$. The $\pm$ subscripts in the right above notations relates to the sign of the real part of the concerned eigenvalues. There holds:

$$sign\left(sin(\theta^+/2)\right)=sign(\tau)\times sign\left(cos(\theta^+/2)\right).$$

We deduce from it that:

$$\mu_- = \frac{1}{2}\lambda - \frac{1}{4}((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(\left(1+\frac{16\tau^2}{(\lambda^2+4\gamma)^2}\right)^{-\frac{1}{2}}+1\right)$$

$$-i\ sign(\tau)\frac{1}{4}((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1-\left(1+\frac{16\tau^2}{(\lambda^2+4\gamma)^2}\right)^{-\frac{1}{2}}\right)$$

and

$$\mu_+ = \frac{1}{2}\lambda + \frac{1}{4}((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(\left(1+\frac{16\tau^2}{(\lambda^2+4\gamma)^2}\right)^{-\frac{1}{2}}+1\right)$$

$$+i\ sign(\tau)\frac{1}{4}((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1-\left(1+\frac{16\tau^2}{(\lambda^2+4\gamma)^2}\right)^{-\frac{1}{2}}\right)$$

Notice that we have:

$$\mu_+|_{(\tau,\gamma)=(0,0)} = \lambda$$

Taking into account that, due to the noncharacteristic boundary assumption, $\lambda \neq 0$, there are two constants C_1 and C_2 such that, for all $\tau \in \mathbb{R}$ and $\gamma > 0$, there holds:

$$\Re e(\mu_+) > C_1 > 0, \qquad \Re e(\mu_-) < C_2 < 0.$$

Indeed, studying the sign of $\Re e(\mu_+^+)$ and $\Re e(\mu_-^-)$ all amounts to the study of the sign of the following expression:

$$2\lambda((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{4}} - sign(\lambda)\left(\lambda^2+4\gamma+((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}}\right),$$

which has the same sign as:

$$sign(\lambda)\left(4\lambda^2((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}} - \left(\lambda^2+4\gamma+((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}}\right)^2\right)$$

$$= -sign(\lambda)\left((\lambda^2+4\gamma)^2+((\lambda^2+4\gamma)^2+16\tau^2)+(8\gamma-2\lambda^2)((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}}\right)$$

Using that $\gamma \geq 0$, we have:

$$(\lambda^2+4\gamma)^2+((\lambda^2+4\gamma)^2+16\tau^2)+(8\gamma-2\lambda^2)((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}}$$

$$\geq (\lambda^2+4\gamma)^2+((\lambda^2+4\gamma)^2+16\tau^2)+(-8\gamma-2\lambda^2)((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}}$$

Noticing that

$$(\lambda^2+4\gamma)^2+((\lambda^2+4\gamma)^2+16\tau^2)+(-8\gamma-2\lambda^2)((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}} = (\lambda^2+4\gamma-((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}})^2 \geq$$

with the equality only holding for $(\tau,\gamma) = 0$, it gives that, if $(\tau,\gamma) \neq (0,0)$:

$$sign(2\lambda((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{4}} - sign(\lambda)\left(\lambda^2+4\gamma+((\lambda^2+4\gamma)^2+16\tau^2)^{\frac{1}{2}}\right)) = -sign(\lambda)$$

Hence we have:

- If $\lambda < 0$, then $\Re e(\mu_+) \geq 0$, with the equality holding only for $(\tau,\gamma) = 0$. Moreover $\Re e(\mu_-) < 0$ for all $(\tau,\gamma) \in \mathbb{R} \times \mathbb{R}^+$.
- If $\lambda > 0$ then $\Re e(\mu_+) > 0$ for all $(\tau,\gamma) \in \mathbb{R} \times \mathbb{R}^+$. In addition, $\Re e(\mu_-) \leq 0$, with the equality holding only for $(\tau,\gamma) = 0$.

$\square$

The same way, the eigenvectors of $\mathbb{A}^-$ associated to the eigenvalues with positive real parts denoted by $(\mu_i^-)_{1\leq i\leq N}$ are given by:

$$\left(\mathbf{w}_i^-\right)_{1\leq i\leq N} := \begin{pmatrix} v_i^- \\ \mu_i^- v_i^- \end{pmatrix}_{1\leq i\leq N}.$$

The eigenvectors of $\mathbb{A}^-$ associated to the eigenvalues with negative real parts denoted by $(\mu_i^-)_{N+1\leq i\leq 2N}$ are given by:

$$\left(\mathbf{w}_i^-\right)_{N+1\leq i\leq 2N} := \begin{pmatrix} v_i^- \\ \mu_{N+i}^- v_i^- \end{pmatrix}_{N+1\leq i\leq 2N}.$$

The family $\left(\mathbf{w}_i^+\right)_{1\leq i\leq N}$ is a basis of $\mathbb{E}_+(\mathbb{A}^-)$. Moreover the $\mu_i^-(\tau,\gamma)$ satisfy:

$$\lambda_j^- = \mu_i^-(\tau,\gamma) - \frac{i\tau+\gamma}{\mu_i^-(\tau,\gamma)}.$$

4.3.2 Expression of an Evans function.

For medium frequencies, that is to say for ζ belonging to a bounded open subset of $\mathbb{R}\times\mathbb{R}^+$ not containing 0, an Evans function is given by:

$$D(\zeta) := \begin{vmatrix} v_1^+ & \dots & v_N^+ & v_1^- & \dots & v_N^- \\ \mu_1^+(\zeta)v_1^+ & \dots & \mu_N^+(\zeta)v_N^+ & \mu_1^-(\zeta)v_1^- & \dots & \mu_N^-(\zeta)v_N^- \end{vmatrix}.$$

For the asymptotic Evans function, when $|\zeta|\to\infty$, we take:

$$\widetilde{D}(\zeta) := \begin{vmatrix} v_1^+ & \dots & v_N^+ & v_1^- & \dots & v_N^- \\ \frac{\mu_1^+(\zeta)}{\Lambda(\zeta)}v_1^+ & \dots & \frac{\mu_N^+(\zeta)}{\Lambda(\zeta)}v_N^+ & \frac{\mu_1^-(\zeta)}{\Lambda(\zeta)}(\hat{\zeta})v_1^- & \dots & \frac{\mu_N^-(\zeta)}{\Lambda(\zeta)}v_N^- \end{vmatrix},$$

Due to its specificity, the asymptotic Evans function for low frequencies will be introduced in the section right below, along with the needed material.

4.3.3 Introduction to a low frequency Evans function.

We will now perform here a detailed analysis of the Evans function for low frequencies. Since some eigenvalues, that we will call hyperbolic, of $\mathbb{A}^{\pm}$ vanishes for $\widetilde{\zeta} = 0$, the associated positive or negative space of $\mathbb{A}^{\pm}$ cease to be well-defined for low frequencies. Although it is the case, we will show we can extend the definition of those spaces in a continuous way. We will later provide explicit computations of those limiting spaces in section 4.3.7. The associated asymptotic Evans function will be computed during section 4.3.8, its nonvanishing meaning that the uniform Evans Condition becomes equivalent to the Evans Condition. The main idea behind our proof is that only the hyperbolic eigenvalues and the associated eigenvectors have to be recomputed for low frequencies. In a first step, we will introduce the appropriate scaling for the low frequency analysis of what corresponds to the hyperbolic block. We recall that $\mathbb{A}^{\pm}$ denotes the following 4×4 sized matrix:

$$\mathbb{A}^{\pm}(\widetilde{\zeta}) := \left(\begin{array}{cc} 0 & Id \\ (i\widetilde{\tau} + \widetilde{\gamma})Id & A^{\pm} \end{array} \right),$$

Moreover, it intervenes in an ODE of the form:

$$\partial_z \left(\begin{array}{c} w^{\pm} \\ \partial_z w^{\pm} \end{array} \right) = \mathbb{A}^{\pm}(\widetilde{\zeta}) \left(\begin{array}{c} w^{\pm} \\ \partial_z w^{\pm} \end{array} \right) + F^{\pm},$$

We have then:

$$\partial_z \left(\begin{array}{c} w^{\pm} \\ \rho^{-1}\partial_z w^{\pm} \end{array} \right) := \left(\begin{array}{cc} 0 & \rho Id \\ \rho^{-1}(i\widetilde{\tau} + \widetilde{\gamma})Id & A^{\pm} \end{array} \right) \left(\begin{array}{c} w^{\pm} \\ \rho^{-1}\partial_z w^{\pm} \end{array} \right) := \rho\check{\mathbb{A}}(\check{\zeta}, \rho) \left(\begin{array}{c} w^{\pm} \\ \rho^{-1}\partial_z w^{\pm} \end{array} \right),$$

where

$$\check{\mathbb{A}}^{\pm}(\check{\zeta}, \rho) := \left(\begin{array}{cc} 0 & Id \\ \rho^{-1}(i\check{\tau} + \check{\gamma})Id & \rho^{-1}A^{\pm} \end{array} \right)$$

with $\check{\tau} := \frac{\widetilde{\tau}}{\rho}$ and $\check{\gamma} := \frac{\widetilde{\gamma}}{\rho}$.

For $\widetilde{\gamma} > 0$,

$$\mathbb{E}_-(\mathbb{A}^+) = \mathbb{E}_-^H(\mathbb{A}^+) \bigoplus \mathbb{E}_-^P(\mathbb{A}^+),$$

where $\mathbb{E}_-^H(\mathbb{A}^+)$ is the space generated by the generalized eigenvectors of $\mathbb{A}^+$ associated to the the hyperbolic eigenvalues of $\mathbb{A}^+$ with negative real part. The same way, $\mathbb{E}_-^P(\mathbb{A}^+)$ stands for the space generated by the generalized eigenvectors of $\mathbb{A}^+$ associated to the the parabolic eigenvalues of $\mathbb{A}^+$ with negative real part. By opposition to the hyperbolic eigenvalues, the parabolic eigenvalues does not cross the imaginary axis even for $\widetilde{\zeta} = 0$. Remark that the dimensions of $\mathbb{E}_-^H(\mathbb{A}^+)$ and $\mathbb{E}_-^P(\mathbb{A}^+)$ are constant. Viewing temporarily $\check{\zeta}$ as a parameter, we introduce the following decomposition:

$$\mathbb{E}_-(\check{\mathbb{A}}^+) = \mathbb{E}_-^H(\check{\mathbb{A}}^+) \bigoplus \mathbb{E}_-^P(\check{\mathbb{A}}^+),$$

like before, we call an eigenvalue of $\check{\mathbb{A}}^+$ hyperbolic if it vanishes for $\check{\zeta} = 0$ an parabolic otherwise. Remark well that, in this case, these denominations are sort of artificial since, by definition, $|\check{\zeta}| = 1$. $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ and $\mathbb{E}_-^P(\check{\mathbb{A}}^+)$ are then defined like before. The extended linear subspace $\mathbb{E}_-^{lim}(\mathbb{A}^+)$ is then given by:

$$\mathbb{E}_-^{lim}(\mathbb{A}^+) = \mathbb{E}_-^H(\check{\mathbb{A}}^+)|_{\check{\tau}=1,\check{\gamma}=0,\rho=0} \bigoplus \mathbb{E}_-^P(\mathbb{A}^+)|_{\zeta=0},$$

where $\mathbb{E}_-^H(\check{\mathbb{A}}^+)|_{\check{\tau}=1,\check{\gamma}=0,\rho=0}$ stands for $\lim_{\check{\gamma}\to 0^+,\check{\tau}^2+\check{\gamma}^2=1} \lim_{\rho\to 0^+} \mathbb{E}_-^H(\check{\mathbb{A}}^+)(\check{\zeta},\rho)$. The same way, $\mathbb{E}_+(\mathbb{A}^-)$ extends continuously to $\mathbb{E}_+^{lim}(\mathbb{A}^-)$ as $\widetilde{\zeta}$ goes to zero, with:

$$\mathbb{E}_+^{lim}(\mathbb{A}^-) = \mathbb{E}_+^H(\check{\mathbb{A}}^-)|_{\check{\tau}=1,\check{\gamma}=0,\rho=0} \bigoplus \mathbb{E}_+^P(\mathbb{A}^-)|_{\zeta=0}.$$

The following Proposition shows the strong interest raised by the ability of computing explicitly $\mathbb{E}_-^{lim}(\mathbb{A}^+)$ and $\mathbb{E}_+^{lim}(\mathbb{A}^+)$.

Proposition 4.3.1. *Let us assume that the $(\widetilde{\mathcal{H}}^\varepsilon, \widetilde{\mathcal{M}})$ satisfies the Evans Condition which means that, for all $\zeta = (\tau,\gamma) \in \mathbb{R}\times\mathbb{R}^+ - \{0_{\mathbb{R}^2}\}$, there holds:*

$$\left|det\left(\widetilde{\mathbb{E}}_-(\mathbb{A}^+(\zeta)), \widetilde{\mathbb{E}}_+(\mathbb{A}^-(\zeta))\right)\right| > 0.$$

Then the four following properties are equivalent:

- $(\widetilde{\mathcal{H}}^\varepsilon, \widetilde{\mathcal{M}})$ *satisfies the* ***Uniform*** *Evans Condition.*
- *There is $\rho_0 > 0$ such that, for all $\zeta = (\tau,\gamma) \in \mathbb{R}\times\mathbb{R}^+ - \{0_{\mathbb{R}^2}\}$, with $|\zeta| < \rho_0$, there holds:*

  $$\left|det\left(\mathbb{E}_-(\mathbb{A}^+(\zeta)), \mathbb{E}_+(\mathbb{A}^-(\zeta))\right)\right| \geq C > 0.$$
- $\left|det\left(\mathbb{E}_-^{lim}(\mathbb{A}^+), \mathbb{E}_+^{lim}(\mathbb{A}^-)\right)\right| > 0.$
- $\mathbb{E}_-^{lim}(\mathbb{A}^+) \bigcap \mathbb{E}_+^{lim}(\mathbb{A}^-) = \{0\}.$

Remark 4.3.2. *If we take $N = 1$ that is to say a scalar system, the uniform Evans condition is always satisfied. As a consequence, the uniform Evans condition also holds if A^+ and A^- are diagonalizable in the same basis.*

4.3.4 Analysis of the medium and high frequencies Evans function for 2×2 systems.

The bases in which A^+ and A^- are diagonal differ in general from each other. However, making the right change of basis, we can always assume that A^- is diagonal without loss of generality. Let us fix a positive real number K, for the Evans condition to hold, it is necessary that, for all $0 < |\zeta| < K$, the real and imaginary part of following determinant do not vanish simultaneously:

$$D(\zeta) = \begin{vmatrix} a & c & 1 & 0 \\ b & d & 0 & 1 \\ a\mu_1^+(\zeta) & c\mu_2^+(\zeta) & \mu_1^-(\zeta) & 0 \\ b\mu_1^+(\zeta) & d\mu_2^+(\zeta) & 0 & \mu_2^-(\zeta) \end{vmatrix}$$

where $\begin{pmatrix} a \\ b \end{pmatrix}$ is the normalized eigenvector associated to λ_1^+, which denotes the smallest eigenvalue of A^+ and $\begin{pmatrix} c \\ d \end{pmatrix}$ is the normalized eigenvector associated to λ_2^+, which is the greatest eigenvalue of A^+. We have thus $a^2 + b^2 = 1$, $c^2 + d^2 = 1$ and $ad - bc \neq 0$. Some computations show that:

$$D(\zeta) = (ad-bc)(\mu_1^+\mu_2^+ + \mu_1^-\mu_2^-) - ad(\mu_1^-\mu_2^+ + \mu_2^-\mu_1^+) + bc(\mu_2^-\mu_2^+ + \mu_1^-\mu_1^+)$$

Notice first that $Im(D(\zeta))$ does vanish for $\tau = 0$, thus a necessary condition in order for the Evans condition to hold is that $\Re e(D(0,\gamma))$ does not vanish for all γ positive. So, We will now study the sign of

$$\Re e D(\zeta) = D_1(\zeta) - D_2(\zeta)$$

where

$$\begin{aligned} D_1(\zeta) := & \, ad(\Re e(\mu_1^+) - \Re e(\mu_1^-))(\Re e(\mu_2^+) - \Re e(\mu_2^-)) \\ & +bc(\Re e(\mu_2^+) - \Re e(\mu_1^-)(\Re e(\mu_2^-) - \Re e(\mu_1^+)). \end{aligned}$$

and

$$D_2(\zeta) := ad(Im(\mu_1^+) - Im(\mu_1^-))(Im(\mu_2^+) - Im(\mu_2^-))$$
$$+bc(Im(\mu_2^+) - Im(\mu_1^-)(Im(\mu_2^-) - Im(\mu_1^+)).$$

Let us denote by $\lambda_1^+ < \lambda_2^+$ the two eigenvalues of A^+ and $\lambda_1^- < \lambda_2^-$ the two eigenvalues of A^-, we have then, for $i \in \{1;2\}$:

$$\mu_i^+ = \frac{1}{2}\lambda_i^+ - \frac{1}{4}((\lambda_i^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(\left(1+\frac{16\tau^2}{\left(\lambda_i^{+2}+4\gamma\right)^2}\right)^{-\frac{1}{2}}+1\right)$$
$$-i\ sign(\tau)\frac{1}{4}((\lambda_i^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1-\left(1+\frac{16\tau^2}{\left(\lambda_i^{+2}+4\gamma\right)^2}\right)^{-\frac{1}{2}}\right)$$

$$\mu_i^- = \frac{1}{2}\lambda_i^- + \frac{1}{4}((\lambda_i^{-2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(\left(1+\frac{16\tau^2}{\left(\lambda_i^{-2}+4\gamma\right)^2}\right)^{-\frac{1}{2}}+1\right)$$
$$+i\ sign(\tau)\frac{1}{4}((\lambda_i^{-2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1-\left(1+\frac{16\tau^2}{\left(\lambda_i^{-2}+4\gamma\right)^2}\right)^{-\frac{1}{2}}\right)$$

As a consequence, restricting ourselves to $\tau = 0$ we have:

$$\mu_i^+|_{\tau=0} = \frac{1}{2}\left(\lambda_i^+ - ((\lambda_i^{+2}+4\gamma)^2)^{\frac{1}{4}}\right)$$

$$\mu_i^-|_{\tau=0} = \frac{1}{2}\left(\lambda_i^- + ((\lambda_i^{-2}+4\gamma)^2)^{\frac{1}{4}}\right).$$

Remark that, because A^+ and A^- are nonsingular, for all positive γ, there holds:

$$\mu_i^+|_{\tau=0} < 0,$$
$$\mu_i^-|_{\tau=0} > 0.$$

However, as γ vanishes, $\mu_i^+|_{\tau=0}$ or $\mu_i^-|_{\tau=0}$ may vanish too depending on the sign of λ_i^+ and λ_i^- .

4.3.5 Some sufficient assumptions for the Evans Condition to hold.

A necessary condition for the uniform Evans condition to hold is that, for all $\gamma > 0$, $|D(0,\gamma)| > 0$, which means that the sign of the following quantity remains strictly the same for all positive γ:

$$Q := ad(\mu_1^+|_{\tau=0} - \mu_1^-|_{\tau=0})(\mu_2^+|_{\tau=0} - \mu_2^-|_{\tau=0})$$
$$+bc(\mu_2^+|_{\tau=0} - \mu_1^-|_{\tau=0})(\mu_2^-|_{\tau=0} - \mu_1^+|_{\tau=0}) := Q_1 + Q_2.$$

For all $\gamma > 0$, we have thus

$$sign(Q_1) = sign(ad)$$

and

$$sign(Q_2) = -sign(bc).$$

Therefore, alternative sufficient conditions in order to obtain $|D(0,\gamma)| > 0, \quad \forall \gamma > 0$ are $sign(ad) = -sign(bc)$ or $ad = 0$ or $bc = 0$. Indeed, as highlighted previously, for all nonzero ζ, $\mu_i^+|_{\tau=0} < 0$ and $\mu_i^-|_{\tau=0} > 0$. Our idea is, restricting ourselves to the cases where $sign(ad) = -sign(bc)$ or $ad = 0$ or $bc = 0$, to search for sufficient conditions on the eigenvalues and eigenvectors of A^+ and A^- in order to ensure that $\Re e(D(\zeta))$ keeps the same sign as $D_1(\zeta)$ for all $\zeta \neq 0$. Take notice that, for all nonzero ζ, $D_1(\zeta)$ keeps strictly the same sign as $D_1|_{\tau=0}(\gamma)$, for $\gamma > 0$. Since $\Re e(D(\zeta)) = D_1(\zeta) - D_2(\zeta)$, if, for some ζ, $D_2(\zeta)$ is of opposite sign of $D_1(\zeta)$, we have to prove that $|D_2(\zeta)| < |D_1(\zeta)|$. The following lemma is useful in the study the sign of $\Re e D(\zeta)$:

Lemma 4.3.3. *Seeing μ^+ and μ^- as two functions of (ζ, λ), for all $\zeta \neq 0$, we have:*

$$Im(\mu^+(\zeta,\lambda)) = Im(\mu^+(\zeta,-\lambda)) = -Im(\mu^-(\zeta,\lambda)) = -Im(\mu^-(\zeta,-\lambda)).$$

Moreover

$$|Im(\mu^+(\zeta,\lambda))| < |\Re e(\mu^+(\zeta,\lambda))|,$$
$$|Im(\mu^-(\zeta,\lambda))| < |\Re e(\mu^-(\zeta,\lambda))|,$$

for all $\tau \neq 0$ and $\gamma \geq 0$.

Proof. The first part of this lemma is trivial, so let us prove the second part. For this purpose, let us fix $\gamma = \gamma_0$, we will then prove by an argument of comparative increasing speed in $|\tau|$ that for all $|\tau| > 0$, we have

$$|Im(\mu^{\pm}(\tau, \gamma_0, \lambda))| < |\Re e(\mu^{\pm}(\tau, \gamma_0, \lambda))|.$$

Let us begin by the study of μ^+. For all γ_0, there holds

$$|\Re e(\mu^+(0, \gamma_0, \lambda))| \geq |Im(\mu^+(0, \gamma_0, \lambda))| = 0,$$

and $|\Re e(\mu^+(\tau, \gamma_0, \lambda))|$, considered as a function of $|\tau|$, is increasing strictly quicker in $|\tau|$ than $|Im(\mu^+(\tau, \gamma_0, \lambda))|$, for all admissible value of (γ_0, λ), which proves the desired result. Indeed, we have:

$$|\Re e(\mu^+)| = -\frac{1}{2}\lambda^+ + \frac{1}{4}((\lambda^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}} + \frac{1}{4}((\lambda^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1+\frac{16\tau^2}{(\lambda^{+2}+4\gamma)^2}\right)^{-\frac{1}{2}}$$

$$|Im(\mu^+)| = \frac{1}{4}((\lambda^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}} - \frac{1}{4}((\lambda^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1+\frac{16\tau^2}{(\lambda^{+2}+4\gamma)^2}\right)^{-\frac{1}{2}}$$

If we fix the growth of $\frac{1}{4}((\lambda^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}$ for increasing $|\tau|$ as a comparison state, the term $\frac{1}{4}((\lambda^{+2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1+\frac{16\tau^2}{(\lambda^{+2}+4\gamma)^2}\right)^{-\frac{1}{2}}$ is accelerating the growth of $|\Re e(\mu^+)|$ as $|\tau|$ gets bigger, but is delaying the growth of $|Im(\mu^+)|$. Noticing that:

$$|\Re e(\mu^-)| = \frac{1}{2}\lambda^- + \frac{1}{4}((\lambda^{-2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}} + \frac{1}{4}((\lambda^{-2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1+\frac{16\tau^2}{(\lambda^{-2}+4\gamma)^2}\right)^{-\frac{1}{2}}$$

$$|Im(\mu^-)| = \frac{1}{4}((\lambda^{-2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}} - \frac{1}{4}((\lambda^{-2}+4\gamma)^2+16\tau^2)^{\frac{1}{4}}\left(1+\frac{16\tau^2}{(\lambda^{-2}+4\gamma)^2}\right)^{-\frac{1}{2}}.$$

Reasoning the same way, we have thus proved that:

$$|Im(\mu^-(\zeta, \lambda))| < |\Re e(\mu^-(\zeta, \lambda))|.$$

□

Theorem 4.3.4. *For $sign(ad) = -sign(bc)$ or $ad = 0$ or $bc = 0$, the Evans condition always holds.*

Proof. We will begin by treating the case of medium frequencies. For $\tau = 0$, it has already been proven that the real part of the Evans function never vanishes and more precisely keeps the sign of ad or $-bc$ (take the non-null one by default). As a direct consequence of lemma 4.3.3, for all $\tau \neq 0$, there holds: $|\tau| > 0$ $\Re e(\mu_2^-) > |Im(\mu_2^-)| > 0$, $-\Re e(\mu_2^+) > |Im(\mu_2^+)| > 0$, $\Re e(\mu_1^-) > |Im(\mu_1^-)| > 0$, $-\Re e(\mu_1^+) > |Im(\mu_1^+)| > 0$. Thus, we have:

$$\Re e(\mu_1^-)\Re e(\mu_2^-) - Im(\mu_1^-)Im(\mu_2^-) \geq \Re e(\mu_1^-)\Re e(\mu_2^-) - |Im(\mu_1^-)||Im(\mu_2^-)| > 0,$$

$$\Re e(\mu_1^-)(-\Re e(\mu_2^+)) + Im(\mu_1^-)Im(\mu_2^+) \geq \Re e(\mu_1^-)(-\Re e(\mu_2^+)) - |Im(\mu_1^-)||Im(\mu_2^+)| > 0,$$

$$(-\Re e(\mu_1^+))\Re e(\mu_2^-) + Im(\mu_1^+)Im(\mu_2^-) \geq (-\Re e(\mu_1^+))\Re e(\mu_2^-) - |Im(\mu_1^+)||Im(\mu_2^-)| > 0,$$

$$(-\Re e(\mu_1^+))(-\Re e(\mu_2^+)) - Im(\mu_1^+)Im(\mu_2^+) \geq (-\Re e(\mu_1^+))(-\Re e(\mu_2^+)) - |Im(\mu_1^+)||Im(\mu_2^+)| > 0.$$

As a consequence, ad has the same sign as:

$$ad(\Re e(\mu_1^-) - \Re e(\mu_1^+))(\Re e(\mu_2^-) - \Re e(\mu_2^+)) - (Im(\mu_1^-) - Im(\mu_1^+))(Im(\mu_2^-) - Im(\mu_2^+)).$$

The same way, for all $\tau \neq 0$, $-bc$ has the same sign as:

$$bc(\Re e(\mu_1^-) - \Re e(\mu_2^+))(\Re e(\mu_1^+) - \Re e(\mu_2^-)) - bc(Im(\mu_1^-) - Im(\mu_2^+))(Im(\mu_1^+) - Im(\mu_2^-)).$$

Hence, assuming $sign(ad) = -sign(bc)$ or $ad = 0$ or $bc = 0$, $\Re e D(\zeta)$ and thus $D(\zeta)$ does not vanish for all nonzero frequencies. The analysis performed here also works for high frequencies, where the eigenvalues $\mu^\pm$ of $\mathbb{A}^\pm$ have to be replaced by $\frac{\mu^\pm}{\Lambda}$, with $\Lambda > 0$, which ends our proof. $\square$

We have proved here Proposition 4.2.8 stated at the beginning of the paper. Remark that this Proposition states that the Evans Condition holds in some cases, without concern for the uniformity.

4.3.6 Some instances for which the uniform Evans condition does not hold.

This section is devoted to the proof of Proposition 4.2.9. We have shown during last section that the Evans condition always holds if $sign(ad) = -sign(bc)$. Consider (a,b,c,d) such that $ad - bc \neq 0$, $a^2 + b^2 = c^2 + d^2 = 1$, and $sign(ad) = sign(bc)$; $\lambda_1^- < \lambda_2^-$, $\lambda_1^+ < \lambda_2^+$. We shall search here for some $(a, b, c, d, \lambda_1^-, \lambda_1^+, \lambda_2^-, \lambda_2^+)$, inducing strong Evans-instabilities. More precisely, we will see that, upon correct choice of these parameters, $D|_{\tau=0}$ can vanish for some positive γ. To construct our example, we begin by making some sign assumptions on the eigenvalues corresponding to $q := \dim \Sigma = 0$:

$$\lambda_1^- < 0, \quad \lambda_2^- > 0, \quad \lambda_1^+ < 0, \quad \lambda_2^+ > 0.$$

For the sake of simplicity, we will assume that a, b, c, d are positive. Denoting by

$$D_a(\gamma) := ad(\Re e(\mu_1^+|_{\tau=0}) - \Re e(\mu_1^-|_{\tau=0}))(\Re e(\mu_2^+|_{\tau=0}) - \Re e(\mu_2^-|_{\tau=0})),$$

$$D_b(\gamma) := bc(\Re e(\mu_2^+|_{\tau=0}) - \Re e(\mu_1^-|_{\tau=0}))(\Re e(\mu_1^+|_{\tau=0}) - \Re e(\mu_2^-|_{\tau=0})),$$

we have $D|_{\tau=0} = D_a - D_b$. Note that $sign(D_a) = sign(D_b)$. Thus, $D|_{\tau=0}$ does not vanish for some $\gamma_0 > 0$ if and only if we have either $D_a > D_b$ for all positive γ, or $D_a < D_b$ for all positive γ. Observe that:

$$D_a(0) = ad(\lambda_1^+ - |\lambda_1^+| - \lambda_1^- - |\lambda_1^-|)(\lambda_2^+ - |\lambda_2^+| - \lambda_2^- - |\lambda_2^-|)$$

$$D_b(0) = bc(\lambda_2^+ - |\lambda_2^+| - \lambda_1^- - |\lambda_1^-|)(\lambda_1^+ - |\lambda_1^+| - \lambda_2^- - |\lambda_2^-|)$$

Due to the assumption we have made on the sign of the eigenvalues, we have:

$$D_a(0) = 4ad|\lambda_1^+||\lambda_2^-|,$$

$$D_b(0) = 0.$$

As a result, by continuity of D_a and D_b with respect to γ, we obtain that $D_a > D_b$ for γ in a positive neighborhood of zero. The interesting fact is that this inequality does not need any strong assumption to hold. Our goal will then be to prove that, for some $\gamma_0 > 0$, we have $D_a < D_b$, by continuity of D_a and D_b with respect to γ, this will prove the existence of a positive γ canceling the Evans function for $\tau = 0$. Remarking that D_a and D_b share some similarities in their constructions, we will take $\lambda_1^+ = -\lambda_2^+$ and $\lambda_1^- = -\lambda_2^-$ in order to build our example. By doing so, we have the simplified expressions of D_a and D_b:

$$D_a = ad\left(8\gamma + 2\sqrt{(\lambda_2^+)^2 + 4\gamma}\sqrt{(\lambda_2^-)^2 + 4\gamma} + 2\lambda_2^+\lambda_2^-\right)$$

$$D_b = bc\left(8\gamma + 2\sqrt{(\lambda_2^+)^2 + 4\gamma}\sqrt{(\lambda_2^-)^2 + 4\gamma} - 2\lambda_2^+\lambda_2^-\right).$$

Now take $bc = 2ad$, ($bc > ad$ would be sufficient to construct the example) denoting by $\gamma_0 := max\left(\frac{(\lambda_2^-)^2}{2}, \frac{(\lambda_2^+)^2}{2}\right)$, there holds $D_b(\gamma_0) > D_a(\gamma_0)$. Indeed,

$$D_b - D_a = bc\left(8\gamma + 2\sqrt{(\lambda_2^+)^2 + 4\gamma}\sqrt{(\lambda_2^-)^2 + 4\gamma} - 6\lambda_2^+\lambda_2^-\right),$$

and $2\sqrt{(\lambda_2^+)^2 + 4\gamma}\sqrt{(\lambda_2^-)^2 + 4\gamma} - 6\lambda_2^+\lambda_2^- \geq 0$ for all $\gamma \geq \gamma_0$. Thus, there is $0 < \gamma_1 < \gamma_0$ such that the Evans function vanishes for $\zeta = (0, \gamma_1)$.

4.3.7 Computation of the extension of the linear subspaces $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ and $\mathbb{E}_+^H(\check{\mathbb{A}}^-)$ in the case A^+ and A^- belongs to $\mathcal{M}_2(\mathbb{R})$.

Let us now inquire on a way to compute $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ and $\mathbb{E}_+^H(\check{\mathbb{A}}^-)$ for 2×2 systems. Due to the symmetry of the problem, we will only investigate the calculus of $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$. For small ρ, corresponding to $\check{\zeta}$ in a neighborhood ω of 0, let us look for an 'hyperbolic' eigenvalue of $\check{\mathbb{A}}^+$ that we will note $\check{\lambda}^+(\check{\zeta}, \rho)$ in a generic manner, and compute its associated eigenvector:

$$\check{\mathbb{A}}^+(\check{\zeta}, \rho)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \check{\lambda}^+\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

Adopting the notation:

$$A^\pm := \begin{pmatrix} a_{11}^\pm & a_{12}^\pm \\ a_{21}^\pm & a_{22}^\pm \end{pmatrix}$$

we get, by multiplying some equations by $\rho > 0$ the following system:

$$\begin{cases} v_3 = \check{\lambda}^+ v_1, \\ v_4 = \check{\lambda}^+ v_2, \\ (i\check{\tau} + \check{\gamma})v_1 + a_{11}^+ v_3 + a_{12}^+ v_4 = \rho\check{\lambda}^+ v_3, \\ (i\check{\tau} + \check{\gamma})v_2 + a_{21}^+ v_3 + a_{22}^+ v_4 = \rho\check{\lambda}^+ v_4 \quad . \end{cases}$$

Making $\rho \to 0^+$ gives then, the following limiting system for low frequencies:

$$\begin{cases} v_3 = \check{\lambda}^+ v_1, \\ v_4 = \check{\lambda}^+ v_2, \\ (i\check{\tau} + \check{\gamma} + a_{11}^+ \check{\lambda}^+) v_1 + a_{12}^+ \check{\lambda}^+ v_2 = 0, \\ a_{21}^+ \check{\lambda}^+ v_1 + (i\check{\tau} + \check{\gamma} + a_{22}^+ \check{\lambda}^+) v_2 = 0 \quad . \end{cases}$$

Take notice that, in the above equation, $\check{\lambda}^+$ is also an unknown. In addition $\check{\lambda}^+ = 0$ is not an eigenvalue since it would imply that $v_1 = v_2 = v_3 = v_4 = 0$. To study the Asymptotic Evans function for low frequency in order to ensure that the Evans Condition holds uniformly, several cases would have to be treated. We will focus here, for some cases, on giving the way to compute the continuous extension of the subspaces to $\gamma = 0$, allowing then to check easily whether the uniform Evans Condition holds or not.
The dimension of the linear subspace $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ is also p_+, the number of negative eigenvalues of A^+. We have then $\mathbb{E}_-^H(\check{\mathbb{A}}^+) = Span\left\{w_1^+, \ldots, w_{p_+}^+\right\}$.

The diagonal case where $a_{12}^+ = 0$ and $a_{21}^+ = 0$.

If $\lambda_j^+ = a_{jj}^+$ is a positive eigenvalue of A^+, then then one of the eigenvectors generating $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ is $\begin{pmatrix} e_j \\ \check{\mu}_j^+ e_j \end{pmatrix}$, where e_j is the j^{th} vector of the canonical basis of $\mathbb{C}^2$ and $\check{\mu}_j^+ = -\frac{i\check{\tau}+\check{\gamma}}{\lambda_j^+}$.

The triangular case where $a_{12}^+ = 0$ and $a_{21}^+ \neq 0$.

$$\begin{cases} v_3 = \check{\lambda}^+ v_1, \\ v_4 = \check{\lambda}^+ v_2, \\ (i\check{\tau} + \check{\gamma} + a_{11}^+ \check{\lambda}^+) v_1 = 0, \\ a_{21}^+ \check{\lambda}^+ v_1 + (i\check{\tau} + \check{\gamma} + a_{22}^+ \check{\lambda}^+) v_2 = 0 \quad . \end{cases}$$

If $\lambda_2^+ = a_{22}^+$ is a positive eigenvalue of A^+, then one of the eigenvectors generating $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ is $\begin{pmatrix} e_2 \\ \check{\mu}_2^+ e_2 \end{pmatrix}$, where e_2 is the second vector of the canonical basis of $\mathbb{C}^2$ and $\check{\mu}_2^+ = -\frac{i\check{\tau}+\check{\gamma}}{\lambda_2^+}$ is one of the eigenvalues with negative real part of $\check{\mathbb{A}}^+$. If $\lambda_1^+ = a_{11}^+$ is a positive eigenvalue of A^+, then $\check{\mu}_1^+ = -\frac{i\check{\tau}+\check{\gamma}}{\lambda_1^+}$ is one of the eigenvalues with negative real part of $\check{\mathbb{A}}^+$. The equation giving the associated eigenvectors is:

$$\left\{\begin{array}{r} v_3 = \check{\mu}_1^+ v_1, \\ v_4 = \check{\mu}_1^+ v_2, \\ v_1 \in \mathbb{C}, \\ v_2 = -\dfrac{a_{21}^+\check{\mu}_1^+}{i\check{\tau}+\check{\gamma}+a_{22}^+\check{\mu}_1^+} v_1 \quad . \end{array}\right.$$

Hence one of the eigenvectors generating $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ is $\begin{pmatrix} i\check{\tau}+\check{\gamma}+a_{22}^+\check{\mu}_1^+ \\ -a_{21}^+\check{\mu}_1^+ \\ \check{\mu}_1^+\left(i\check{\tau}+\check{\gamma}+a_{22}^+\check{\mu}_1^+\right) \\ -a_{21}^+\left(\check{\mu}_1^+\right)^2 \end{pmatrix}$.

The triangular case where $a_{12}^+ \neq 0$ and $a_{21}^+ = 0$.

This case behaves similarly to the other triangular case just treated. If $\lambda_1^+ = a_{11}^+$ is a positive eigenvalue of A^+, then we can take

$$w_1^+ = \begin{pmatrix} e_1 \\ \check{\mu}_1^+ e_1 \end{pmatrix}$$

where e_1 is the first vector of the canonical basis of $\mathbb{C}^2$ and $\check{\mu}_1^+ = -\frac{i\check{\tau}+\check{\gamma}}{\lambda_1^+}$. If $\lambda_2^+ = a_{22}^+$ is a positive eigenvalue of A^+, then one of the eigenvectors generating $\mathbb{E}_-^H(\check{\mathbb{A}}^+)$ is $\begin{pmatrix} -a_{12}^+\check{\mu}_2^+ \\ i\check{\tau}+\check{\gamma}+a_{11}^+\check{\mu}_2^+ \\ -a_{12}^+\left(\check{\mu}_2^+\right)^2 \\ \check{\mu}_2^+\left(i\check{\tau}+\check{\gamma}+a_{11}^+\check{\mu}_2^+\right) \end{pmatrix}$, where $\check{\mu}_2^+ = -\frac{i\check{\tau}+\check{\gamma}}{\lambda_2^+}$ is one of the eigenvalues with negative real part of $\check{\mathbb{A}}^+$.
These computations will allow us to conclude quickly the proof Proposition 4.2.10 done next section.

4.3.8 End of the proof of Proposition 4.2.10.

In view of the results proved until this section, we only lack the proof of the **uniform** nonvanishing of the Evans function as the frequencies come in a neighborhood of zero. For the examples given in Proposition 4.2.10, modulo a change of basis, we take:

$$A^- := \begin{pmatrix} d_1^- & 0 \\ 0 & d_2^- \end{pmatrix},$$

$$A^+ := \begin{pmatrix} d_1^+ & \alpha \\ 0 & d_2^+ \end{pmatrix}$$

where d_1^-, d_2^-, d_1^+, d_2^+ and α are such that:
$\alpha \neq 0$ $d_1^- < 0$, $d_1^+ > 0$, $d_1^- \neq d_2^-$, and $d_1^+ \neq d_2^+$. Following Proposition 4.2.10 we will split our low frequency analysis of the Evans function into three parts depending on the signs of d_2^- and d_2^+.

The case $d_2^- < 0$ and $d_2^+ > 0$.

Note first that we are now considering a completely outgoing or expansive case, which implies that all the eigenvalues of $\mathbb{A}^+$ and $\mathbb{A}^-$ are hyperbolic. The computation of the asymptotic Evans function for low frequencies need the extension of the linear subspaces $\mathbb{E}_-(\mathbb{A}^+)$ and $\mathbb{E}_+(\mathbb{A}^-)$, which ceases to be well-defined as $|\zeta| \to 0$. Our problem satisfies our stability assumption (Uniform Evans Condition) iff the function D_{low} does not vanish for $\check{\gamma} = 0, \check{\tau} = 1$. D_{low} is defined as the modulus of the following determinant:

$$\begin{vmatrix} 1 & -\alpha\check{\mu}_2^+ & 1 & 0 \\ 0 & \nu_2^+ & 0 & 1 \\ \check{\mu}_1^+ & -\alpha(\check{\mu}_2^+)^2 & \check{\mu}_1^- & 0 \\ 0 & \check{\mu}_2^+\nu_2^+ & 0 & \check{\mu}_2^- \end{vmatrix}$$

We have thus:

$$D_{low} = |\nu_2^+||\check{\mu}_2^- - \check{\mu}_2^+||\check{\mu}_1^- - \check{\mu}_1^+|,$$

from which we get, since $|i\check{\tau} + \check{\gamma}| = 1$, that:

$$D_{low} = \left|1 - \frac{d_1^+}{d_2^+}\right| \left|-\frac{1}{d_2^-} + \frac{1}{d_2^+}\right| \left|-\frac{1}{d_1^-} + \frac{1}{d_1^+}\right| > 0.$$

Note well that, surprisingly D_{low} does not even depend of $\check{\zeta}$.

The case $d_2^- < 0$ and $d_2^+ < 0$.

We proceed like we have just done in the case where $d_2^- < 0$ and $d_2^+ > 0$. This time, thanks to the sign of d_2^+, $\mathbb{A}^+$ has one hyperbolic eigenvalue with negative real part that we will note $\check{\mu}_1^+$ and one parabolic eigenvalue with negative real part that we will note $\check{\mu}_2^+$. $\check{\mu}_1^+$ vanishes for $\widetilde{\zeta} = 0$, whereas $\check{\mu}_2^+|_{\widetilde{\zeta}=0} = d_2^+$. $\check{\mathbb{A}}^+$ has two eigenvalues with negative real parts:

$$\check{\mu}_1^+(\check{\zeta}) = -\frac{i\check{\tau} + \check{\gamma}}{d_1^+},$$

$$\check{\mu}_2^+(\check{\zeta}) = d_2^+.$$

As a consequence, we get that our problem satisfies our stability assumption (Uniform Evans Condition) iff the function D_{low} does not vanish for $\check{\gamma} = 0, \check{\tau} = 1$. D_{low} is defined as the modulus of the following determinant:

$$\begin{vmatrix} 1 & \alpha & 1 & 0 \\ 0 & d_2^+ - d_1^+ & 0 & 1 \\ \check{\mu}_1^+ & d_2^+\alpha & \check{\mu}_1^- & 0 \\ 0 & d_2^+(d_2^+ - d_1^+) & 0 & \check{\mu}_2^- \end{vmatrix}$$

We have thus:

$$D_{low} = |\check{\mu}_1^- - \check{\mu}_1^+||d_2^+ - d_1^+||\check{\mu}_2^- - d_2^+|,$$

from which we get, since $|i\check{\tau} + \check{\gamma}| = 1$, that:

$$D_{low} = \left| -\frac{1}{d_1^-} + \frac{1}{d_1^+} \right| \left| d_2^+ - d_1^+ \right| \left(\left(\frac{\check{\tau}}{d_2^-} \right)^2 + \left(\check{\gamma} + d_2^+ \right)^2 \right).$$

Hence $D_{low}|_{\check{\tau}=1,\check{\gamma}=0} > 0$.

The case $d_2^- > 0$ and $d_2^+ > 0$.

This time $\check{\mathbb{A}}^+$ has two eigenvalues with negative real parts:

$$\check{\mu}_1^+(\check{\zeta}) = -\frac{i\check{\tau} + \check{\gamma}}{d_1^+},$$

$$\check{\mu}_2^+(\check{\zeta}) = -\frac{i\check{\tau} + \check{\gamma}}{d_2^+}.$$

As a consequence, we get that our problem satisfies our stability assumption (Uniform Evans Condition) iff the function D_{low} does not vanish for $\check{\gamma} = 0$ and $\check{\tau} = 1$. D_{low} is defined as the modulus of the following determinant:

$$\begin{vmatrix} 1 & -\alpha\check{\mu}_2^+ & 1 & 0 \\ 0 & \nu_2^+ & 0 & 1 \\ \check{\mu}_1^+ & -\alpha(\check{\mu}_2^+)^2 & \check{\mu}_1^- & 0 \\ 0 & \check{\mu}_2^+\nu_2^+ & 0 & \check{\mu}_2^- \end{vmatrix}$$

We have thus:

$$D_{low} = |\nu_2^+||\check{\mu}_1^- - \check{\mu}_1^+||\check{\mu}_2^- - \check{\mu}_2^+|;$$

hence, since $|i\check{\tau} + \check{\gamma}| = 1$, we obtain:

$$D_{low} = |i\check{\tau} + \check{\gamma} + d_1^+\check{\mu}_2^+||\check{\mu}_1^- - \check{\mu}_1^+||d_2^- - \check{\mu}_2^+|$$

and then

$$D_{low} = \left|1 - \frac{d_1^+}{d_2^+}\right|\left|-\frac{1}{d_1^-} + \frac{1}{d_1^+}\right|\left(\left(d_2^- + \frac{\check{\gamma}}{d_2^+}\right)^2 + \left(\frac{\check{\tau}}{d_2^+}\right)^2\right) > 0.$$

Partie II:

Approximation de Solutions de Problèmes aux Limites Hyperboliques par des Méthodes de Pénalisation de Domaine.

Chapter 5

Pénalisation de problèmes semi-linéaires symétriques hyperboliques avec des conditions au bord dissipatives

Ce chapitre reprend le papier [FG07] intitulé "Penalization approach of semi-linear symmetric hyperbolic problems with dissipative boundary conditions", co-écrit avec Olivier Guès, soumis à publication en Juillet 2007.

Abstract

In this paper, we introduce a penalization method in order to approximate the solutions of the initial boundary value problem for a semi-linear first order symmetric hyperbolic system, with dissipative boundary conditions. The penalization is carefully chosen in order that the convergence to the wished solution is sharp, does not generate any boundary layer, and applies to fictitious domains.

5.1 Introduction

In this paper we consider the initial boundary value problem for a symmetric first order hyperbolic system ([Fri58]), with maximally strictly dissipative boundary conditions, on a characteristic boundary. Typically the problem writes

$$
(5.1.1) \qquad \begin{cases} Lu = F(t,x,u) \text{ in }]0,T[\times\Omega \\ u_{|]0,T[\times\partial\Omega} \in \mathcal{N} \\ u_{t=0} = 0. \end{cases}
$$

where Ω is a suitably regular open set of $\mathbb{R}^d$ with smooth boundary, L is the first order symmetric hyperbolic system, $\mathcal{N}$ is a smooth bundle on $\mathbb{R}\times\partial\Omega$ defining the boundary conditions, and F a smooth map that can be non linear.

The subject of the paper is mainly motivated by numerical analysis: we want to approximate the solution u of (5.1.1) by the solution v^ε of a well chosen *Cauchy problem* (instead of a boundary value problem), where the complementary part of Ω will be penalized, using a large parameter $1/\varepsilon$:

$$
(5.1.2) \qquad \begin{cases} L^\sharp v^\varepsilon + \frac{1}{\varepsilon}\mathbf{1}_{\mathbb{R}\times\Omega^c} M v^\varepsilon = F^\sharp(t,x,v^\varepsilon) \text{ in }]0,T[\times\mathbb{R}^d \\ u_{|t=0} = 0. \end{cases}
$$

where $L^\sharp$, $F^\sharp$ are extensions of L and F to the whole space $\mathbb{R}\times\mathbb{R}^d$, and $M(t,x)$ is a suitable symmetric and ≥ 0 matrix. We solve this problem under general assumptions, the main point being the existence of the matrix M. We give two solutions for the matrix M.

1/ The first solution is a positive definite matrix which was introduced by J. Rauch [Rau79] in the study of the linear case, related to the work by J. Rauch an C. Bardos [BR82] on singular perturbations. We show that for this approach v^ε converges to u, and that a boundary layer forms close to $\partial\Omega$, on the side of Ω^c. This is Theorem 5.2.6. In the linear non-characteristic case, the occurrence of boundary layers has been already observed in [Dro97].

2/ The second solution contains an improvement of the previous one, and in this case the matrix M is no more invertible. In this approach the convergence of v^ε is better because there are *no boundary layers at all, at any order*. This result is stated in Theorem 5.2.7. In the two results, the key point is the use of Rauch's matrix ([Rau79]).

Let us also mention other interesting features of our results:

3/ Still motivated by concrete applications we show that one can chose the operator $L^\sharp$ in a such a way that, instead of solving the Cauchy problem (5.1.2), one needs only to solve the problem

$$\left\{\begin{array}{c} L^\sharp v^\varepsilon + \frac{1}{\varepsilon}\mathbf{1}_{\mathbb{R}\times\Omega^c} M v^\varepsilon = F^\sharp(t,x,v^\varepsilon) \text{ in }]0,T[\times\Omega^\sharp \\ u_{|\{t=0\}\times\Omega^\sharp} = 0, \end{array}\right. \tag{5.1.3}$$

with *no boundary conditions on* $\mathbb{R}\times\partial\Omega^\sharp$, where $\Omega^\sharp$ is an open set containing Ω. No regularity is assumed on $\Omega^\sharp$ and it can be a polyhedral domain. When Ω is bounded, $\Omega^\sharp$ can be taken bounded. This is the subject of the Corollary 5.2.8.

4/ If $u \in H^\infty([0,T]\times\Omega)$, the convergence of v^ε towards u will hold on $[0,T]\times\Omega$.

In the paper, to simplify the proof, we treat the case where Ω is a half space $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d, x_d > 0\}$, but we give the extension to the general case in a short section, without proof. We also restrict ourselves to the case where $u_{t<0} = 0$ in order to avoid the problem of compatibility conditions for the Cauchy problem. The section 2 is devoted to the precise statement of the assumptions and results in the case $\Omega = \mathbb{R}^n_+$. The section 3 describes the extension to the practical case when Ω is bounded. Section 4 is devoted to the proof of Theorem 5.2.7. Section 5 contains the proof of theorem 5.2.6.

5.2 Main results

Let us consider a symmetric hyperbolic operator

$$L = A_0(t,x)\partial_t + \sum_{j=1}^{d} A_j(t,x)\partial_j + B(t,x)$$

where A_j, $j = 0,\cdots,d$ and B are $N\times N$ real matrices defined on $\mathbb{R}^{1+d}_+ := \{x \in \mathbb{R}^d | x_d > 0\}$ ($\mathbb{R}^{1+d}_-$ is defined by $\mathbb{R}^{1+d}_- := \{x \in \mathbb{R}^d | x_d < 0\}$). We assume that all the entries of A_j, $j = 0,\cdots,d$ and B are in $\mathcal{C}^\infty_b(\mathbb{R}^{1+d})$, the set of smooth functions bounded with bounded derivatives of all order. We also assume that all the matrices are constant out of a bounded subset of $\mathbb{R}^{1+d}$ and that for $j = 0,\cdots d$, A_j is symmetric, A_0 being uniformly positive definite on $\mathbb{R}^{1+d}$.

Assumption 5.2.1. *The matrix $A_d(t,y,0)$ has a constant rank on $\mathbb{R}^d$.*

When the rank of A_d is N, the boundary is non characteristic for L, and when $rank A_d < N$, the boundary is *characteristic of constant multiplicity*. In the sequel of the paper, if M is a symmetric $N \times N$ matrix we will note:

$$\mathbb{E}_+\big(M\big) = \sum_{\lambda>0} \ker(M-\lambda I),$$

and

$$\mathbb{E}_-\big(M\big) = \sum_{\lambda<0} \ker(M-\lambda I).$$

Let us call d_+ the dimension of $\mathbb{E}_+\big(A_d(t,y,0)\big)$, which is independent of (t,y) and is also the number of > 0 eigenvalues of A_d (counted with their multiplicities). For all $T > 0$ we note

$$\Omega_T :=]-1,T[\times\mathbb{R}^d, \quad \Omega_T^+ = \Omega_T \cap \{x_d > 0\}, \quad \Omega_T^- = \Omega_T \cap \{x_d < 0\}$$

and $\Gamma_T := \{(t,x) \in \mathbb{R}^{1+d} \mid -1 < t < T, \quad x_d = 0\}$. We are interested in the initial boundary value problem in $\{x_d > 0\}$ with boundary conditions

$$u \in \mathcal{N}(t,y) \tag{5.2.1}$$

where $\mathcal{N}(t,y)$ defines a smooth vector bundle on the boundary. Let us denote by $\mathbb{P}_0(t,y)$ the orthogonal projector of $\mathbb{R}^N$ onto $\mathrm{Ker}A_d(t,y,0)$.

Assumption 5.2.2. *$\mathcal{N}(t,y)$ is a linear subspace of $\mathbb{R}^N$ depending smoothly on $(t,y) \in \mathbb{R}^d$ with $\dim \mathcal{N}(t,y) = N - d_+$, and there exists a constant $c_0 > 0$ such that for all $v \in \mathbb{R}^N$ and all $(t,y) \in \mathbb{R}^d$:*

$$v \in \mathcal{N}(t,y) \;\Rightarrow\; \langle A_d(t,y,0)v, v\rangle \leq -c_0 \|\big(I - \mathbb{P}_0(t,y)\big)v\|^2. \tag{5.2.2}$$

This kind of boundary conditions (5.2.1) were introduced by K. O. Friedrichs and are called 'maximally strictly dissipative' (see [Maj84]). Let us consider a smooth mapping $F \in \mathcal{C}^\infty(\mathbb{R}^{1+d+N} : \mathbb{R}^N)$, such that for all $\alpha \in \mathbb{N}^{1+d+N}$, $\partial^\alpha_{t,x,u} F$ is bounded on $\mathbb{R}^{1+d} \times K$ for any compact $K \subset \mathbb{R}^N$, such that $F(t,x,0) \in H^\infty(\mathbb{R}^{1+d})$ and $F_{|t<0} = 0$.

It follows from the results of [Rau85] and [Guè90] that there exist $T_0 > 0$ and a unique $u \in H^\infty(\Omega^+_{T_0})$ such that

$$\left\{\begin{array}{l} Lu = F(t,x,u) \text{ in } \Omega^+_{T_0}, \\ u(t,y,0) \in \mathcal{N}(t,y) \text{ on } \Gamma_0\ , \\ u_{|\Gamma_0} = 0. \end{array}\right. \tag{5.2.3}$$

From now on, the real $T_0 > 0$ and $u \in H^\infty(\Omega^+_{T_0})$ are fixed once for all.

Let us now describe our main result. We want to approximate u by the solution of a singularly perturbed Cauchy problem in the domain Ω_{T_0}, where the subdomain $\Omega^-_{T_0}$ is penalized. The first step is to extend the operator L to an operator defined on $-\infty < x_d < +\infty$. Let us consider an operator

$$L^\sharp := \sum_{j=0}^{j=d} A^\sharp_j(t,x)\partial_j + B^\sharp(t,x) \tag{5.2.4}$$

where the matrices $A^\sharp_j$ and $B^\sharp$ are $N \times N$ and are defined on $\mathbb{R}^{1+d}$ and coincide with the matrices A_j and B if $x_d > 0$. We assume that the restrictions $A^\sharp_{j|x_d\le 0}$ and $B^\sharp_{|x_d\le 0}$ are in $C^\infty(\overline{\mathbb{R}^{1+d}_-})$, constant outside a bounded subset of $\overline{\mathbb{R}^{1+d}_-}$. For all $(t,x) \in \mathbb{R}^{1+d}$, the matrices $A^\sharp_j(t,x)$ $j = 0,\dots,d$ are symmetric and $A^\sharp_0(t,x)$ is uniformly positive definite on $\mathbb{R}^{1+d}$. An important point is that we assume continuity of $A^\sharp_d$:

Assumption 5.2.3. *The matrix $A^\sharp_d$ is continuous on $\mathbb{R}^{1+d}$.*

Hence the matrices $A^\sharp_j$ are allowed to be discontinuous across $\{x_d = 0\}$ excepted for $A^\sharp_d$. For example on $\{x_d < 0\}$, one can take $A^\sharp_0 = I$, $A_j = 0$ for $j = 1\cdots, d-1$, $B^\sharp = 0$, and such an extension is obtained by taking simply

$$L^\sharp := \partial_t + A_d(t,y,0)\partial_d \text{ on } \{x_d < 0\}, \tag{5.2.5}$$

which satisfies our assumptions.

Now the problem is to find a matrix $M(t,x)$ with $\mathcal{C}^\infty$ coefficients, constant out of a compact set in $\mathbb{R}^{1+d}$, such that the hyperbolic Cauchy problem

$$\left\{\begin{array}{l} L^\sharp v^\varepsilon + \frac{1}{\varepsilon}\mathbf{1}_{\{x_d<0\}} M v^\varepsilon = \mathbf{1}_{\{x_d>0\}}\, F(t,x,v^\varepsilon) \text{ in } \Omega_T \\ v^\varepsilon_{|\Gamma_0} = 0. \end{array}\right. \tag{5.2.6}$$

admits a unique piecewise smooth solution v^ε which converges to u in $\Omega^+_{T_0}$, as $\varepsilon > 0$ goes to 0. First of all, it's worth emphasizing that the problem (5.2.6) makes sense, although the matrices $A^\sharp_j$, $j = 0, \cdots, d-1$, could be *discontinuous* across the hyperplane $\{x_d = 0\}$. The point is that A_d is continuous so that one can write the principal part of the differential operator in a conservative form. We refer to the appendix for a more detailed discussion of this point, with a proof of the well-posedness.

We will give two solutions to this problem of penalization. Let us begin with a preliminary lemma.

Lemma 5.2.4. *For all point $p = (t, y, 0) \in \mathbb{R}^d$ there exist a neighborhood $\mathcal{V}(p)$ in $\mathbb{R}^{1+d}$ and $\Psi \in C^\infty\big(\mathcal{V}, \mathrm{GL}_N(\mathbb{R})\big)$ satisfying*

$$0 < c \leq |\det \Psi(t, x)| < c^{-1}, \quad \forall (t, x) \in \mathcal{V}$$

for some constant c, and such that: $\mathbb{E}_+\big(\Psi^t A_d \Psi\big)^\perp = \Psi^{-1}\mathcal{N}$ on $\{x_d = 0\} \cap \mathcal{V}$.

Proof. Let us note $\widetilde{A_d} := \Psi^t A_d \Psi$ and $\mathbb{E}_{\leq 0}\big(\widetilde{A_d}\big) = \mathbb{E}_+\big(\widetilde{A_d}\big)^\perp$, which can also be defined by

$$\mathbb{E}_{\leq 0}\big(\widetilde{A_d}\big) = \sum_{\lambda \leq 0} \ker(\widetilde{A_d} - \lambda I).$$

There holds $\mathbb{E}_{\leq 0}\big(\widetilde{A_d}\big) = \Psi^{-1}\mathbb{E}_{\leq 0}\big(\Psi\, \Psi^t A_d\big)$. Hence the claimed result is equivalent to find Ψ which satisfies $\mathcal{N}(t, y) = \mathbb{E}_{\leq 0}\big(\Psi\, \Psi^t A_d\big)$ for all $x = (t, y, 0) \in \mathbb{R}^d \times \{0\}$. Now, we know from a lemma by J. Rauch in [Rau79] that there exists a smooth symmetric definite and positive matrix $E(t, y)$ such that $\mathcal{N}(t, y) = \mathbb{E}_{\leq 0}\big(E(t, y) A_d(t, y, 0)\big)$ which concludes the proof by taking $\Psi = OE^{1/2}$, where O is any orthogonal matrix. As the proof shows, Lemma 5.2.4 is nothing but Rauch's result, expressed in a different way. Note that Rauch's result has been extended recently by F. Sueur ([Sue05]) to the case of general dissipative boundary conditions (not necessarily *strictly* dissipative). □

To simplify the presentation we will make the following assumption which enables one to use *only one* mapping Ψ to reduce the problem, as we will see in section 5.4.1. Nevertheless, this assumption is not a real restriction, as explained in the following comment.

Assumption 5.2.5. *We assume that one can take $\mathcal{V}(p) = \mathbb{R}^{1+d}$ in Lemma 5.2.4.*

Let us fix for all the sequel a mapping Ψ as in the lemma with $\mathcal{V}(p) = \mathbb{R}^{1+d}$.

Example 5.2.1. *We show that in the general case when the Assumption 5.2.5 is not satisfied, one can introduce an extended system (by using a suitable partition of unity) which satisfies the assumption* 5.2.5. *By compactness there exist a finite number of open set $\mathcal{V}_1, \cdots, \mathcal{V}_k$ with the corresponding functions of Lemma 5.2.4 $\Psi_j \in C^\infty\big(\mathcal{V}_j, GL_N(\mathbb{R})\big)$, $j = 1, \cdots, k$ and associated cut-off functions $\chi_j \in C_0^\infty(\mathbb{R}^{d+1})$ $j = 1, \cdots, k$ such that $\mathrm{supp}\chi_j \subset \mathcal{V}_j$ and $\sum \chi_j = 1$ in $\mathbb{R}^{d+1}$. For all $j \in \{1, \cdots, k\}$ the function $U_j(t,x) := \chi_j(t,x)u(t,x)$ satisfies the following system where we have noted for $(t,x) \in \mathbb{R}^{1+d}, \xi \in \mathbb{R}^{1+d}$, $L(t,x;\xi) = \sum_0^d \xi_i A_i$:*

$$
\left\{
\begin{array}{l}
LU_j = L(t,x;d\chi_j)u + \chi_j F(t,x,u) \text{ in } \Omega^+_{T_0}, \\
U_j(t,y,0) \in \mathcal{N}(t,y) \text{ on } \Gamma_{T_0}\ , \\
U_{j|\Gamma_0} = 0.
\end{array}
\right.
\tag{5.2.7}
$$

It follows that the function

$$U(t,x) := \big(U_1(t,x), \cdots, U_k(t,x)\big)$$

satisfies a larger $kN \times kN$ hyperbolic system

$$
\left\{
\begin{array}{l}
\mathbb{L}U = \mathbb{F}(t,x,U) \text{ in } \Omega^+_{T_0}, \\
U(t,y,0) \in \mathbb{B}(t,y) \text{ on } \Gamma_{T_0}\ , \\
U_{|\Gamma_0} = 0,
\end{array}
\right.
\tag{5.2.8}
$$

where $\mathbb{B} = \mathcal{N} \times \cdots \times \mathcal{N}$,

$$
\mathbb{F}(t,x,U) = \begin{pmatrix} \vdots \\ L(t,x;d\chi_j) \sum_j U_j + \chi_j F(t,x,\sum_j U_j) \\ \vdots \end{pmatrix}
\tag{5.2.9}
$$

and

$$
\mathbb{L}U = \begin{pmatrix} LU_1 \\ \vdots \\ LU_k \end{pmatrix}.
$$

This is again an semi-linear symmetric hyperbolic system with maximally dissipative boundary conditions, satisfying the assumptions 5.2.1, 5.2.2 and the assumption 5.2.5.

We will state two theorems. The second one is 'better' than the first one, because it gives a kind of sharp result as long as one is interested in the quality of the convergence of v^ε towards u. Let us begin by introducing the matrix

$$R(t,x) := (\Psi^{-1})^t(t,x)\,\Psi^{-1}(t,x), \quad \forall (t,x) \in \mathbb{R}^{1+d}, \tag{5.2.10}$$

which is, for fixed (t,x), a symmetric and uniformly positive definite matrix (it is the matrix introduced by Rauch in [Rau79], and used in the proof of Lemma 5.2.4). The matrix R gives a good answer to the problem of penalization and our first result concerns the following problem:

$$\begin{cases} L^\sharp w^\varepsilon + \frac{1}{\varepsilon}\mathbf{1}_{\{x_d<0\}} R w^\varepsilon = \mathbf{1}_{\{x_d>0\}}\, F(t,x,w^\varepsilon) \text{ in } \Omega_T \\ w^\varepsilon_{|\Gamma_0} = 0. \end{cases} \tag{5.2.11}$$

Theorem 5.2.6. *There is $\varepsilon_0 > 0$ such that, for all $\varepsilon \in]0,\varepsilon_0]$, the problem* (5.2.11) *has a unique solution $w^\varepsilon \in L^2(\Omega_{T_0}) \cap L^\infty(\Omega_{T_0})$ on Ω_{T_0}. Moreover $w^\varepsilon_{|\Omega^\pm_{T_0}} \in H^\infty(\Omega^\pm_{T_0})$ and the following estimates hold*

$$\|u - w^\varepsilon_{|\Omega^+_{T_0}}\|_{H^s(\Omega^+_{T_0})} = O(\varepsilon),\ \forall s \in \mathbb{R}, \tag{5.2.12}$$

and

$$\|w^\varepsilon_{|\Omega^-_{T_0}}\|_{H^s(\Omega^-_{T_0})} = O(\varepsilon^{-s+\frac{1}{2}}),\ \forall s \in \mathbb{R}, \tag{5.2.13}$$

Let us comment on two points. First, we insist on the fact that the solution w^ε converges to u **on the whole set** $\Omega^+_{T_0}$ which was fixed in the preliminaries and where u was defined. Second, the convergence in (5.2.12) holds for all s, which means that there is no singularity with respect to ε in $\Omega^+_{T_0}$, although the perturbation is singular. However, the estimates (5.2.13) indicates that the behavior of w^ε is singular (with respect to ε) in $\Omega^-_{T_0}$. Indeed, the proof of the theorem gives a more precise result, and shows the existence of a boundary layer in the domain $\Omega_{T_0}^-$, and more precisely an asymptotic expansion (we note $y = (x_1, \cdots, x_{d-1})$):

$$w^\varepsilon(t,x) = W(t,y,x_d/\varepsilon) + 0(\varepsilon) \tag{5.2.14}$$

where $W(t,y,z)$ is a boundary layer profile in the sense that

$$\lim_{z\to-\infty} W(t,y,z) = 0$$

(see section 5.5). This is very natural since the problem is a singular perturbation problem. Furthermore, this is not surprising since boundary layers already appeared in the work by J. Droniou devoted to the linear noncharacteristic case ([Dro97]), which can be seen as a special case of our result.

Let us now state our second result. Denote by $\mathbb{P}$ the orthogonal projector of $\mathbb{R}^N$ onto $(\Psi^{-1}\mathcal{N})^\perp$, and note

$$M(t,x) = (\Psi^t)^{-1}(t,x)\mathbb{P}(t,x)\Psi^{-1}(t,x) \tag{5.2.15}$$

which is a symmetric matrix, depending smoothly of $(t,x) \in \mathbb{R}^{1+d}$.

Theorem 5.2.7. *Let us chose the matrix M defined by* (5.2.15)*. There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in]0,\varepsilon_0]$, the problem* (5.2.6) *has a unique solution $v^\varepsilon \in L^2(\Omega_{T_0}) \cap L^\infty(\Omega_{T_0})$ on Ω_{T_0}. Moreover $v^\varepsilon_{|\Omega^\pm_{T_0}} \in H^\infty(\Omega^\pm_{T_0})$ and*

$$\|u - v^\varepsilon_{|\Omega^+_{T_0}}\|_{H^s(\Omega^+_{T_0})} = O(\varepsilon),\ \forall s \in \mathbb{R}. \tag{5.2.16}$$

The restriction of v^ε to $\Omega^-_{T_0}$ also converges in $L^2(\Omega^-_{T_0})$ towards a function noted u^- defined on $\Omega^-_{T_0}$. More precisely, the following estimate holds:

$$\|u^- - v^\varepsilon_{|\Omega^-_{T_0}}\|_{H^s(\Omega^-_{T_0})} = O(\varepsilon),\ \forall s \in \mathbb{R}.$$

Moreover, the behavior of v^ε is not singular with respect to ε in the sense that

$$\|\partial^\alpha v^\varepsilon_{|\Omega^\pm_{T_0}}\|_{L^2(\Omega^\pm_{T_0})} = O(1), \quad \forall \alpha \in \mathbb{N}^{1+d}. \tag{5.2.17}$$

In that case, the convergence holds on each side $\Omega^{\pm}_{T_0}$ in $H^s(\Omega^{\pm}_{T_0})$ respectively for all $s \in \mathbb{R}$, which means that there is no singularity with respect to ε, although the perturbation is singular. In particular there are no boundary layers (at any order) and **the convergence rate is optimal**. In $\Omega^{+}_{T_0}$ the limit of v^ε is u, and in $\Omega^{-}_{T_0}$ the limit of v^ε is a smooth function in $H^\infty(\Omega^{-}_{T_0})$ which is described precisely in the section 5.4.

The theorem 5.2.7 is proved in section 5.4. The theorem 5.2.12 is proved in section 5.5.

Application: fictive boundary and absorbing layer. In practice one has a lot of freedom in the choice of the matrices $A^\sharp_j(t,x)$ in $x_d < 0$ as we have already said. A very interesting choice for numerical applications is to chose this matrices in order that *all the eigenvalues of $A^\sharp_d(t,x)$ are < 0 when $x_d = -\delta$ for some fixed $\delta > 0$.* For example, one can take

(5.2.18)

$$A^\sharp_d(t,y,x_d) = \left(1 + \frac{x_d}{\delta}\right) A_d(t,y,0) + \frac{x_d}{\delta}\mathrm{Id}, \quad \text{for } -\delta < x_d < 0, t \in]-1, T_0[$$

and

$$A^\sharp_j(t,x) = 0, \quad \text{for } x_d < 0, \quad j = 1, \cdots, d-1. \tag{5.2.19}$$

Then, instead of considering the Cauchy problem (5.2.6), one introduces the domain

$$\Omega^\sharp_{T_0} := \{(t,x) \in \mathbb{R}^{1+d} | -1 < t < T_0, -\delta < x_d\}$$

and consider the problem

$$\begin{cases} L^\sharp v^\varepsilon + \frac{1}{\varepsilon}\mathbf{1}_{\{x_d<0\}} M v^\varepsilon = \mathbf{1}_{\{x_d>0\}}\, F(t,x,v^\varepsilon) \text{ in } \Omega^\sharp_{T_0} \\ v^\varepsilon_{|\Gamma_0} = 0. \end{cases} \tag{5.2.20}$$

for which *no boundary condition is needed* precisely because $A_{d|x_d=-\delta}$ is negative definite.

Corollary 5.2.8. *There exist $\varepsilon_0 > 0$ such that, for all $\varepsilon \in]0, \varepsilon_0]$, the problem* (5.2.20) *has a unique solution $v^\varepsilon \in L^2(\Omega^\sharp_{T_0}) \cap L^\infty(\Omega^\sharp_{T_0})$. Furthermore, $v^\varepsilon_{|\pm x_d>0} \in H^\infty\big(\Omega^\sharp_{T_0} \cap \{\pm x_d > 0\}\big)$, and $v^\varepsilon_{|\Omega^+_{T_0}} \to u$ in $H^s(\Omega_{T_0^+})$ for all $s \in \mathbb{R}$ as $\varepsilon \to 0$ and*

$$\|\partial^\alpha v^\varepsilon_{|x_d<0}\|_{L^2\left(\Omega^\sharp_{T_0}\cap\{x_d<0\}\right)} = O(\varepsilon).$$

Proof. The problem (5.2.20) is well posed since the boundary $\{x_d = -\delta\}$ is a maximal strictly negative subspace for $A_d^\sharp(t, y, -\delta)$. On the other hand, the restriction to $\Omega_{T_0}^\sharp$ of the solution of the Cauchy problem (5.2.6) is a solution of (5.2.20). Hence, the two solutions coincide and the result is a consequence of the Theorem 5.2.7. $\square$

This result shows that it is enough to solve the problem without any boundary condition on $\Omega_{T_0}^\sharp$ and for ε small enough this will give a good approximation (up to $0(\varepsilon)$) of the solution u of the original mixed problem. The region $-\delta < x_d < 0$ is a layer which 'absorbs' the energy of the outgoing waves is such a way that the behavior of the solution in the domain $\{x_d > 0\}$ is arbitrarily close to that of u, as ε goes to 0.

5.3 More general domains

The result of this paper can be easily extended to the case of more general domains. For example, if Ω is a bounded connected open subset of $\mathbb{R}^d$, with smooth boundary $\partial\Omega$, and locally on one side of $\partial\Omega$, the two theorems can be extended to the mixed problem in $[0, T] \times \Omega$. In that case the matrix $A_d(t, y, 0)$ has to be replaced by the matrix

$$\mathcal{A}(t, x) := \sum_{i=1}^{i=d} \nu_i(x) A_i(t, x)$$

where $\nu(x) = \big(\nu_1(x), \cdots, \nu_d(x)\big)$ is the outgoing unitary normal at $x \in \partial\Omega$. Instead of Assumption 5.2.1, we assume that $\mathcal{A}$ has a constant rank on $\mathbb{R} \times \partial\Omega$, and d_+ denotes the constant dimension of $\mathbb{E}_+\big(\mathcal{A}(x)\big)$, $x \in \partial\Omega$. Instead of assumption 5.2.2, we assume that $\mathcal{N}$ is a real $\mathcal{C}^\infty$ bundle on $\partial\Omega$ of dimension $N - d_+$, and that for every $x \in \partial\Omega$, the quadratic form $v \mapsto \langle \mathcal{A}(t, x)v, v\rangle$ is definite negative on $\mathcal{N}(x) \cap \ker \mathcal{A}(x)^\perp$. In the Lemma 5.2.4, the conclusion is that

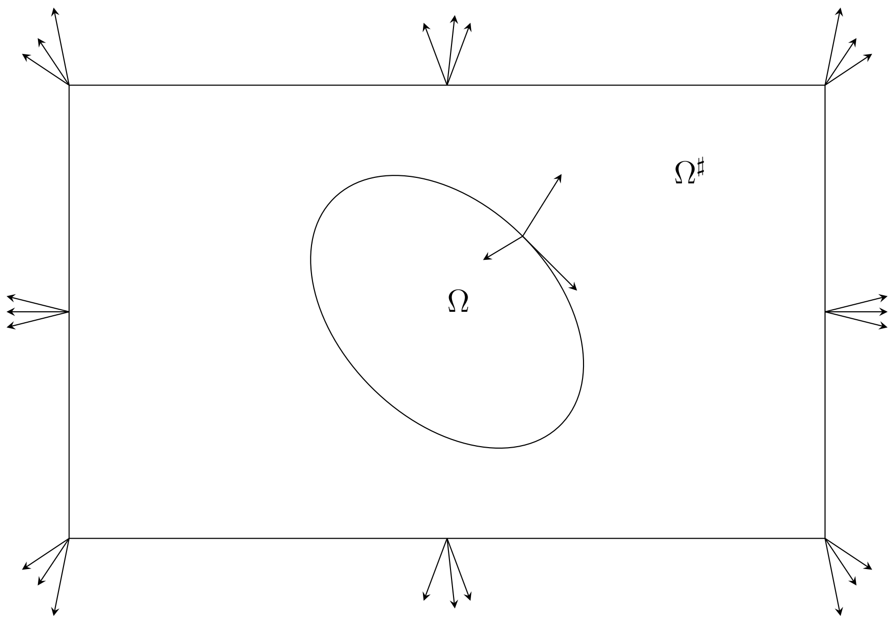
(5.3.1)
$$\mathbb{E}_+\big(\Psi^t(t, x)\mathcal{A}(t, x)\Psi(t, x)\big)^\perp = \Psi(t, x)^{-1}\mathcal{N}(x), \quad \forall (t, x) \in (\mathbb{R}\times\partial\Omega)\cap\mathcal{V}(p).$$

where $\Psi \in \mathcal{C}^\infty(\mathcal{V}(p))$, for $p \in \mathbb{R} \times \partial\Omega$. Finally, Instead of the Assumption 5.2.5 one assumes on can take$\mathcal{V}$ containing $\mathbb{R} \times \Omega$ (or simply

$]0, T_0[\times\Omega$ which is enough). Concerning the extension $L^\sharp$ of the operator to the exterior of $\mathbb{R} \times \Omega$ one requires that the extension of $\mathcal{A}$

$$\mathcal{A}^\sharp(t, x) = \sum_{i=1}^{d} \nu_i(x) A_i^\sharp(t, x)$$

is *continuous* across $\mathbb{R} \times \partial\Omega$ (and this corresponds to the Assumption 5.2.3). In particular, one can take an extension on a thin neighborhood of $\mathbb{R} \times \Omega$ of the form $\mathbb{R} \times \Omega^\sharp$. If one chooses $\Omega^\sharp$ to be a regular open set with completely outgoing eigenvectors for $\mathcal{A}^\sharp(t, x)$ when $x \in \partial\Omega^\sharp$, we will have again to solve a Cauchy problem in $[0, T_0] \times \Omega^\sharp$ (without boundary conditions on $\partial\Omega^\sharp$) and the solution u^ε will converges in $[0, T_0] \times \Omega$ to the solution u of the mixed problem in Ω with boundary conditions $\mathcal{N}$. The set $\Omega^\sharp \setminus \Omega$ plays the role of an 'absorbing layer' which enables one to completely forget about the boundary conditions on $\partial\Omega$, while still solving a problem on the *bounded* domain $\Omega^\sharp$. As a matter of fact, having in mind numerical applications, it is interesting to emphasize the fact that one can take for $\Omega^\sharp$ a polyhedral domain, with boundaries parallel to the coordinate axes for example.



The picture: on the boundary of $\partial\Omega$ the characteristic modes can be ingoing, outgoing, or tangent to $\partial\Omega$. But on the boundary of the extended domain $\Omega^\sharp$, all the fields are outgoing: hence none boundary condition is needed.

5.4 Proof of Theorem 5.2.7

5.4.1 Step 1: reduction of the problem

From now on, to simplify the notations, we will forget the symbol $\sharp$ of the extended matrices to $x_d < 0$, and simply note A_j, B, and L instead of $A_j^\sharp$, $B^\sharp$, $L^\sharp$. There is no risk of confusion since, initially, all the matrices were not defined for $x_d < 0$.

The main idea in the proof of Theorem 5.2.7 is to change the unknown in order to replace the problem (5.2.3) by a new (equivalent) one of the same type, but where the space $\mathcal{N}$ is exactly the subspace orthogonal to $\mathbb{E}_+\big(A_d(t,y,0)\big)$ in $\mathbb{R}^N$.

Let us consider a matrix $\Phi(t,x)$, $N \times N$, with entries C^∞ on $\mathbb{R}^{1+d}$, constant outside a compact set, and such that

$$0 < c \leq |\det \Phi(t,x)| < c^{-1}, \quad \forall (t,x) \in \mathbb{R}^{1+d}$$

for some constant c. Let us define $\widetilde{u}(t,x) := \Phi^{-1}(t,x)u(t,x)$, which satisfies the new system

$$\begin{cases} \widetilde{L}\widetilde{u} = \widetilde{F}(t,x,\widetilde{u}) \text{ in } \Omega^+_{T_0}, \\ \\ \widetilde{u}_{|\Gamma_{T_0}} \in \widetilde{\mathcal{N}} \text{ on } \Gamma_{T_0}, \\ \\ \widetilde{u}_{|\Omega^+_0} = 0. \end{cases} \tag{5.4.1}$$

where $\widetilde{L} = \sum \widetilde{A_j}\partial_j + \widetilde{B}$ with $\widetilde{A_j} := \Phi^t A_j \Phi$, $\widetilde{B} = \Phi^t B\Phi + \Phi^t L(\Phi)$, $\widetilde{F} = \Phi^t F$, and $\widetilde{\mathcal{N}}(t,y) := \Phi^{-1}(t,y,0)\mathcal{N}(t,y)$. The new system (5.4.1) is still a symmetric hyperbolic system which satisfies the same assumptions 5.2.1 and 5.2.2.

From now on, we fix a function Φ as in Lemma 5.2.4. We can define our penalization matrix on the formulation (5.4.1). Let us introduce the orthogonal projector $\widetilde{\mathbb{P}}_+(t,x)$ of $\mathbb{R}^N$ onto $\mathbb{E}_+\big(\widetilde{A_d}(t,x)\big)$, $\widetilde{\mathbb{P}}_-(t,x)$ the orthogonal projector onto $\mathbb{E}_-\big(\widetilde{A_d}(t,x)\big)$, and $\widetilde{\mathbb{P}}_0(t,x)$ the orthogonal projector onto $\ker\big(\widetilde{A_d}(t,x)\big)$:

$$I = \widetilde{\mathbb{P}}_+(t,x) + \widetilde{\mathbb{P}}_-(t,x) + \widetilde{\mathbb{P}}_0(t,x), \quad \forall (t,x) \in \mathbb{R}^{d+1}. \tag{5.4.2}$$

Taking $\Phi := \Psi$, the boundary condition of Equation (5.4.1) becomes:

$$\widetilde{u}_{|\Gamma_{T_0}} \in \Big(\mathbb{E}_+(\widetilde{A}_d)\Big)^\perp$$

We will show that a solution of the problem is given by considering the following Cauchy problem

$$\begin{cases} \widetilde{L}\widetilde{v}^\varepsilon + \varepsilon^{-1}\mathbf{1}_{\{x_d<0\}}\widetilde{\mathbb{P}}_+\widetilde{v}^\varepsilon = \mathbf{1}_{\{x_d>0\}}\,\widetilde{F}(t,x,\widetilde{v}^\varepsilon) \text{ in } \Omega_{T_0}, \\ \widetilde{v}^\varepsilon_{|\Omega_0} = 0. \end{cases} \tag{5.4.3}$$

and that the solution $\widetilde{v}^\varepsilon$ of (5.4.3) exists on Ω_{T_0}, and converges to $\widetilde{u}$. Going back to the original unknown $u = \Psi v$, this will prove the main result.

5.4.2 Step 2: an approximate solution

In this section we construct an approximate solution of (5.4.3) of the form

$$\widetilde{v}^\varepsilon_a(t,x) = \sum_{j=0}^{M} \varepsilon^j \widetilde{V}_j(t,x), \tag{5.4.4}$$

where the $\widetilde{V}_j \in H^\infty(\Omega_{T_0})$ for all $j = 0, \cdots, M$. In order to solve the problem (5.4.3) we solve the equation in the half space $x_d > 0$ and in $x_d < 0$ coupled with the transmission condition

$$[(Id - \widetilde{\mathbb{P}}_0)\widetilde{v}^\varepsilon]_{\{x_d=0\}} = 0. \tag{5.4.5}$$

This transmission condition (5.4.5) splits into the following two equations:

$$\widetilde{\mathbb{P}}_+[\widetilde{v}^\varepsilon]_{\{x_d=0\}} = 0 \quad , \quad \widetilde{\mathbb{P}}_-[\widetilde{v}^\varepsilon]_{\{x_d=0\}} = 0. \tag{5.4.6}$$

In general, if $v(t,x)$ is a function defined on Ω_{T_0} we will note $v^+ := v_{|x_d>0}$ and $v^- := v_{|x_d<0}$. Substituting $\widetilde{v}_a^\varepsilon$ of the form in 5.4.4 for $\widetilde{v}^\varepsilon$ in 5.4.3 gives, at the order ε^{-1}, and in $x_d < 0$,

$$\widetilde{\mathbb{P}}_+\widetilde{V}_0^- = 0 \quad \text{in } \Omega_{T_0}^- \tag{5.4.7}$$

which implies by the first equation of (5.4.6) that

$$\big(\widetilde{\mathbb{P}}_+\widetilde{V}_0^+\big)_{|\Gamma_{T_0}} = 0 \quad \text{on } \Gamma_{T_0}. \tag{5.4.8}$$

On the side $x_d > 0$, at the order ε^0 one gets the equation $\widetilde{L}\widetilde{V}_0^+ = \widetilde{F}(t,x,\widetilde{V}_0^+)$. Therefore, $\widetilde{V}_0^+$ is defined as the unique solution of the mixed problem

$$\begin{cases} \widetilde{L}\widetilde{V}_0^+ = \widetilde{F}(t,x,\widetilde{V}_0^+) \text{ in } \Omega_{T_0}, \\ \big(\widetilde{\mathbb{P}}_+\widetilde{V}_0^+\big)_{|\Gamma_{T_0}} = 0 \quad \text{on } \Gamma_{T_0}, \\ \widetilde{V}^0_{+|\Omega_0} = 0. \end{cases} \tag{5.4.9}$$

and by uniqueness, $\widetilde{V}_0^+ = \widetilde{u}$ as desired. On the side $x_d < 0$ the term of order ε^0 is

$$\widetilde{L}\widetilde{V}_0^- + \widetilde{\mathbb{P}}_+\widetilde{V}_1^- = 0. \tag{5.4.10}$$

Let us call $\widetilde{\Pi}(t,x) := \widetilde{\mathbb{P}}_-(t,x) + \widetilde{\mathbb{P}}_0(t,x) = (I - \widetilde{\mathbb{P}}_+)$. Since $\widetilde{\mathbb{P}}_+\widetilde{V}_0^- = 0$, there holds $\widetilde{V}_0^- = \widetilde{\Pi}\widetilde{V}_0^-$, and applying $\widetilde{\Pi}$ on the left to Equation (5.4.10) leads to

$$\widetilde{\Pi}\,\widetilde{L}\,\widetilde{\Pi}\,(\widetilde{\Pi}\widetilde{V}_0^-) \;=\; 0 \quad \text{in } \Omega_{T_0}^-, \tag{5.4.11}$$

where $\widetilde{\Pi}\,\widetilde{L}\,\widetilde{\Pi}$ is a *symmetric hyperbolic operator* acting on the space

$$\mathcal{E} := \{\, u \in L^2(\Omega^-_{T_0}, \mathbb{R}^N) \,|\, (I - \widetilde{\Pi})u = 0 \,\}, \tag{5.4.12}$$

that is on the space of functions polarized on $\ker \widetilde{\mathbb{P}}_+$. For instance, this kind of hyperbolic operator appears naturally in the context of weakly non linear geometric optics (see [MJR99]) where it is a usual tool. Now the second part of the transmission relations (5.4.6) can be written

$$\widetilde{\mathbb{P}}_- \widetilde{\Pi} \widetilde{V}^-_{0\ |x_d=0} = (\widetilde{\mathbb{P}}_- \widetilde{V}^+_0)_{|x_d=0} \tag{5.4.13}$$

which is a boundary condition for the unknown $\widetilde{\Pi}\widetilde{V}^-_0$ because $\widetilde{V}^+_0$ in the right hand side is already known ($\widetilde{V}^+_0 = \widetilde{u}$). This boundary condition for $\widetilde{\Pi}V^-_0$ is maximally dissipative for the operator $\widetilde{\Pi}\,\widetilde{L}\,\widetilde{\Pi}$, hence $\widetilde{\Pi}\widetilde{V}^-_0 = \widetilde{V}^-_0$ is uniquely defined by (5.4.11), (5.4.13), and the initial condition $(\widetilde{\Pi}\widetilde{V}^-_0)_{|\Omega^-_0} = 0$. Since the problem is linear, $\widetilde{V}^-_0$ is actually defined on $\Omega^-_{T_0}$.

Going back to Equation (5.4.10) shows that $\widetilde{\mathbb{P}}_+\widetilde{V}^-_1$ is determined (as was $\widetilde{\mathbb{P}}_+\widetilde{V}^-_0$ by the ε^{-1} terms),

$$\widetilde{\mathbb{P}}_+ \widetilde{V}^-_1 = -\widetilde{\mathbb{P}}_+ \widetilde{L} \widetilde{V}^-_0, \tag{5.4.14}$$

and Equation (5.4.10) is now entirely satisfied.

The construction follows by induction. For example, let us continue the construction in order to determine $\widetilde{V}_1$ completely. The equation for the ε^1 terms in the side $\{x_d > 0\}$ is

$$\widetilde{L}\widetilde{V}^+_1 = \widetilde{F}'_u(t, x, \widetilde{V}^+_0)\widetilde{V}^+_1, \tag{5.4.15}$$

and the functions $\widetilde{V}^+_1$ and $\widetilde{V}^-_1$ are linked by the transmission conditions (5.4.6) which writes at the order ε^1:

$$\widetilde{\mathbb{P}}_+ \widetilde{V}^+_{1\ |\Gamma_{T_0}} = \widetilde{\mathbb{P}}_+ \widetilde{V}^-_{1\ |\Gamma_{T_0}}, \tag{5.4.16}$$

and

$$\widetilde{\mathbb{P}}_- \widetilde{V}^-_{1\ |\Gamma_{T_0}} = \widetilde{\mathbb{P}}_- \widetilde{V}^+_{1\ |\Gamma_{T_0}}. \tag{5.4.17}$$

Since the right hand side of (5.4.16) is known (from (5.4.14)), the function $\widetilde{V}^+_1$ is the unique solution of the well posed mixed problem (5.4.15), (5.4.16), with the initial condition $\widetilde{V}^+_{1\ |t<0} = 0$.

It remains to determine $(\mathrm{Id} - \widetilde{\mathbb{P}}_+)\widetilde{V}_1^- = \widetilde{\Pi}\widetilde{V}_1^-$. The equation for the terms of order ε^1 in $\{x_d < 0\}$ is

$$\widetilde{L}\widetilde{V}_1^- + \widetilde{\mathbb{P}}_+\widetilde{V}_2^- = 0. \tag{5.4.18}$$

We first apply $\widetilde{\Pi}$ to the equation in order to cancel the term in $\widetilde{V}_2^-$ which is unknown, and replace $\widetilde{V}_1^- = \widetilde{\Pi}\widetilde{V}_1^- + \widetilde{\mathbb{P}}_+\widetilde{V}_1^- = \widetilde{\Pi}\widetilde{V}_1^- - \widetilde{\mathbb{P}}_+\widetilde{L}\widetilde{V}_0^-$ which leads to the equation

$$\widetilde{\Pi}\,\widetilde{L}\,\widetilde{\Pi}\,(\widetilde{\Pi}\widetilde{V}_1^-) \;=\; \widetilde{\Pi}\widetilde{L}\widetilde{\mathbb{P}}_+\widetilde{L}\widetilde{V}_0^- \quad \text{in } \Omega_{T_0}^-. \tag{5.4.19}$$

This is again a symmetric hyperbolic system in the space $\mathcal{E}$ and Equation (5.4.17) appears to be a boundary condition for this system since the right hand side of (5.4.17) is known. Hence $\widetilde{\Pi}\widetilde{V}_1^-$ is the unique solution of the mixed problem (5.4.19), (5.4.17), with the initial condition $\widetilde{\Pi}\widetilde{V}_1^-{}_{|t<0} = 0$. In conclusion, $\widetilde{V}_1$ is completely determined, and going back to Equation (5.4.18) we see that $\widetilde{\mathbb{P}}_+\widetilde{V}_2^-$ is also determined, and that Equation (5.4.18) is entirely satisfied. The next steps of the construction are completely analogous.

5.4.3 Step 3: estimations

We have constructed an approximate solution $\widetilde{v}_a^\varepsilon$ of the form (5.4.4) of the problem (5.4.3), satisfying

$$\left\{\begin{array}{l} \widetilde{L}\widetilde{v}_a^\varepsilon + \varepsilon^{-1}\mathbf{1}_{\{x_d<0\}}\widetilde{\mathbb{P}}_+\widetilde{v}_a^\varepsilon = \mathbf{1}_{\{x_d>0\}}\,\widetilde{F}(t,x,\widetilde{v}_a^\varepsilon) + \varepsilon^k r^\varepsilon \text{ in } \Omega_{T_0}, \\ \widetilde{v}_{a|\Omega_0}^\varepsilon = 0. \end{array}\right. , \tag{5.4.20}$$

where the error term r^ε is piecewise smooth:

$$\|r^\varepsilon\|_{L^2(\Omega_{T_0})} = O(1), \quad \|r^\varepsilon_{|\pm x_d>0}\|_{H^m(\Omega^\pm_{T_0})} = O(1), \quad (\varepsilon \to 0^+, \forall m \in \mathbb{N}). \tag{5.4.21}$$

We look for an exact solution $\widetilde{v}^\varepsilon$ of the form

$$\widetilde{v}^\varepsilon = \widetilde{v}_a^\varepsilon + \varepsilon\widetilde{w}^\varepsilon \tag{5.4.22}$$

where $\widetilde{w}^\varepsilon$ is defined by the system

$$\left\{\begin{array}{l} \widetilde{L}\widetilde{w}^\varepsilon + \varepsilon^{-1}\mathbf{1}_{\{x_d<0\}}\widetilde{\mathbb{P}}_+\widetilde{w}^\varepsilon = \mathbf{1}_{\{x_d>0\}}\,\widetilde{G}(t,x,\widetilde{v}_a^\varepsilon,\varepsilon\widetilde{w}^\varepsilon)\widetilde{w}^\varepsilon + \varepsilon^{k-1}r^\varepsilon \text{ in } \Omega_{T_0}, \\ \widetilde{w}^\varepsilon{}_{|\Omega_0} = 0. \end{array}\right.$$

where $\widetilde{G}$ is the C^∞ function defined by the Taylor formula:

$$\widetilde{G}(t,x,v,\varepsilon w)w = \varepsilon^{-1}\big(\,\widetilde{F}(t,x,v+\varepsilon w) - \widetilde{F}(t,x,v)\,\big). \tag{5.4.23}$$

This is a semi-linear hyperbolic system, and we will solve it by using a standard Picard's iterative scheme, where we note $\mathbf{1}_- = \mathbf{1}_{\{x_d<0\}}$ and $\mathbf{1}_+ = \mathbf{1}_{\{x_d>0\}}$:

(5.4.24)
$$\begin{cases} \widetilde{L}\widetilde{w}^{\varepsilon,\nu+1} + \varepsilon^{-1}\mathbf{1}_-\widetilde{\mathbb{P}}_+\widetilde{w}^{\varepsilon,\nu+1} - \mathbf{1}_+\,\widetilde{G}(t,x,\widetilde{v}_a^\varepsilon,\varepsilon\widetilde{w}^{\varepsilon,\nu})\widetilde{w}^{\varepsilon,\nu+1} = \varepsilon^{k-1}r^\varepsilon, \\ \widetilde{w}^{\varepsilon,\nu+1}{}_{|\Omega_0} = 0. \end{cases}$$

In order to show the convergence of the sequence $\widetilde{w}^{\varepsilon,\nu}$ we need estimations for the following linear problem

$$\begin{cases} \widetilde{L}\mathbf{v} + \varepsilon^{-1}\mathbf{1}_-\widetilde{\mathbb{P}}_+\mathbf{v} - \mathbf{1}_+\widetilde{G}(\widetilde{v}_a^\varepsilon,\varepsilon\mathbf{b})\mathbf{v} = \varepsilon^{k-1}r^\varepsilon, \\ \mathbf{v}_{|\Omega_0} = 0, \end{cases} \tag{5.4.25}$$

where $\widetilde{G}(\widetilde{v}^\varepsilon,\varepsilon\mathbf{b}) = \widetilde{G}(t,x,\widetilde{v}^\varepsilon,\varepsilon\mathbf{b})$ where $\mathbf{b}$ is a given function, which plays the role of $\widetilde{w}^{\varepsilon,\nu}$ when solving the system for the unknown $\mathbf{v} = \widetilde{w}^{\varepsilon,\nu+1}$.

Let us introduce some notations. We will denote by Z_j the vector fields $Z_j = \partial_j$ if $j = 0,\dots,d-1$ and $Z_d := \frac{x_d}{<x_d>}\partial_d$, with $<x_d> = (1+x_d^2)^{1/2}$. We will note $Z^\alpha := Z_0^{\alpha_0}Z_1^{\alpha_1}\cdots Z_d^{\alpha_d}$ for $\alpha = (\alpha_0,\cdots,\alpha_d) \in \mathbb{N}^{1+d}$. For $\lambda > 0$ we will note

$$\|v\|_{0,\lambda} := \Big(\,\int_{\Omega_{T_0}} e^{-2\lambda t}|v(t,x)|^2 dtdx\,\Big)^{1/2}$$

and for $m \in \mathbb{N}$ and $\lambda > 0$

$$\|v\|_{m,\lambda,\varepsilon} := \sum_{|\alpha|\le m} \lambda^{m-|\alpha|}\|(\sqrt{\varepsilon}Z)^\alpha v\|_{0,\lambda}.$$

We will also need the following norms, corresponding to the same definition but where the domain of integration is replaced by $\Omega_{T_0}^-$:

$$\|\mathbf{v}\|_{m,\lambda,\varepsilon,-} := \sum_{|\alpha|\le m} \lambda^{m-|\alpha|}\|e^{-\lambda t}(\sqrt{\varepsilon}Z)^\alpha \mathbf{v}\|_{L^2(\Omega_{T_0}^-)},$$

and $\|.\|_{m,\lambda,\varepsilon,+}$ the norm where $\Omega_{T_0}^-$ is replaced by $\Omega_{T_0}^+$. We denote by $H_{co}^m(\Omega_{T_0})$ the subspace of all $\mathbf{v} \in L^2(\Omega_{T_0})$ such that $\|\mathbf{v}\|_{m,\lambda,\varepsilon}$ is finite. We will also note $|u|_\infty := \|u\|_{L^\infty(\Omega_{T_0})}$.

Proposition 5.4.1. *Let G be the function defined in* (5.5.21). *For all* $\mathbf{b}, \mathbf{v} \in H^m_{co}(\Omega_{T_0}) \cap L^\infty(\Omega_{T_0})$ *valued in* $\mathbb{R}^N$*, the function* $\mathbf{1}_{\{x_d>0\}} G(t, x, \widetilde{v}^\varepsilon_a, \varepsilon \mathbf{b})\mathbf{v}$ *is also in* $H^m_{co}(\Omega_{T_0}) \cap L^\infty(\Omega_{T_0})$. *Moreover, for all $R > 0$ there exists $C(R) > 0$ such that, if $\|\mathbf{b}\|_{L^\infty(\Omega_{T_0})} \leq R$ there holds:*
(5.4.26)

$$\|\mathbf{1}_{\{x_d>0\}} G(t, x, \widetilde{v}^\varepsilon_a, \varepsilon \mathbf{b})\mathbf{v}\|_{m,\lambda,\varepsilon} \leq C(R)\big(\|\mathbf{v}\|_{m,\lambda,\varepsilon} + |\mathbf{v}|_\infty \, \|\mathbf{b}\|_{m,\lambda,\varepsilon}\big),$$

for all $\varepsilon > 0$.

Proof. This is a 'Moser' type estimate, which follows in a classical way from the corresponding weighted Gagliardo-Nirenberg estimates proved in the appendix of [Guè90], and the Hölder estimate. □

Let us introduce a new notation. We will denote by $H^1_\pm(\Omega_T)$ the space of functions $u \in L^2(\Omega_T)$ such that $u_{|\Omega_T^+} \in H^1(\Omega_T^+)$ and $u_{|\Omega_T^-} \in H^1(\Omega_T^-)$. We can now prove the following estimate on the linear problem:

Proposition 5.4.2. *Let $R > 0$ and $m \in \mathbb{N}$. There are constants $c_m(R) > 0$ and $\lambda_m(R) > 1$ such that the following holds true. For all $\mathbf{b} \in H^m_{co}(\Omega_{T_0}) \cap L^\infty(\Omega_{T_0})$ such that $|\mathbf{b}|_\infty \leq R$, for all $\mathbf{f} \in H^m_{co}(\Omega_{T_0}) \cap H^1_\pm(\Omega_{T_0})$, with $\mathbf{f}_{|t<0} = 0$, the problem* (5.4.25) *has a unique solution $\mathbf{v} \in H^m_{co}(\Omega_{T_0}) \cap H^1_\pm(\Omega_{T_0})$. Moreover, it follows that $\mathbf{v} \in L^\infty(\Omega_{T_0})$ and the following estimate holds*
(5.4.27)

$$\lambda^{1/2}\|\mathbf{v}\|_{m,\lambda,\varepsilon} + \varepsilon^{-1/2}\,\|\widetilde{\mathbb{P}}\,\mathbf{v}_-\|_{m,\lambda,\varepsilon,-} \leq c_m(R)\,\lambda^{-1/2}\,\big(\|\mathbf{f}\|_{m,\lambda,\varepsilon} + \|\mathbf{b}\|_{m,\lambda,\varepsilon}|\mathbf{v}|_\infty\big).$$

for all $\lambda \geq \lambda_m(R)$, and all $\varepsilon > 0$.

Proof. In all the proof we will note $\widetilde{\mathbb{P}}$ instead of $\widetilde{\mathbb{P}}_+$ to simplify.

1/ The first step is the L^2 estimate. Let us call $\mathbf{f}$ the right hand side of Equation (5.4.25), and note $\widetilde{\mathbf{v}} = e^{-\lambda t}\mathbf{v}$ so that $\mathbf{v}$ satisfies

$$\left\{\begin{array}{l} \widetilde{L}\widetilde{\mathbf{v}} + \lambda\widetilde{\mathbf{v}} + \varepsilon^{-1}\mathbf{1}_{\{x_d<0\}}\widetilde{\mathbb{P}}\,\widetilde{\mathbf{v}} = e^{-\lambda t}\mathbf{f}, \\ \widetilde{\mathbf{v}}_{|\Omega_0} = 0, \end{array}\right. \tag{5.4.28}$$

Taking the scalar product of the equations with $\widetilde{\mathbf{v}}$ and integrating by parts in Ω_{T_0} leads to the following L^2 estimate where we note $\mathbf{v}_- = \mathbf{v}_{|\Omega^-_{T_0}}$:

$$\lambda^{1/2}\|\mathbf{v}\|_{0,\lambda} + \frac{1}{\varepsilon^{1/2}}\|\widetilde{\mathbb{P}}\,\mathbf{v}_-\|_{0,\lambda,-} \lesssim \frac{1}{\lambda^{1/2}}\|\mathbf{f}\|_{0,\lambda}, \tag{5.4.29}$$

for all $\lambda \geq \lambda_0$.

2/ In order to estimate higher order derivatives, we need to prepare the system and change again of unknown. Since the spaces $\mathbb{E}_+\big(A_d(t,y,0)\big)$, $\mathbb{E}_-\big(A_d(t,y,0)\big)$ and $\ker A_d(t,y,0)$ have constant rank, there exist for all $(t^0,y^0) \in \mathbb{R}^d$ a neighborhood $\mathcal{V}(t^0,y^0)$ of (t^0,y^0) in $\mathbb{R}^d$ and a smooth matrix $\Psi(t,y) \in \mathcal{C}^\infty\big(\mathcal{V}(t^0,y^0), \mathcal{M}_{N\times N}(\mathbb{R})\big)$, $\Psi^t\Psi = \mathrm{Id}$ such that $\Psi(t,y)\mathbb{E}_+(t,y,0)$, $\Psi(t,y)\mathbb{E}_-(t,y,0)$ and $\Psi(t,y)\ker A_d(t,y,0)$ are constant linear subspaces of $\mathbb{R}^N$ (=independant of (t,y)). Let us note these spaces respectively: $\mathbb{V}_-$, $\mathbb{V}_+$ and $\mathbb{V}_0$. To simplify the proof, we also assume that one can take $\mathcal{V}(0,0) = \mathbb{R}^d$, so that one has just to work with one change of variable. (In the general case, one would have to introduce a finite number of local coordinate patches). We introduce the unknown defined by $\mathbf{u}(t,x) := \Psi(t,y)\mathbf{v}(t,x)$. The system satisfied by $\mathbf{u}$ is

$$\partial_t \mathbf{u} + \sum_1^d \Psi A_j \Psi^t \partial_j \mathbf{u} + \frac{1}{\varepsilon}\mathbf{1}_{\{x_d<0\}} \Psi \mathbb{P}\, \Psi^t \mathbf{u} + \mathcal{B}(\varepsilon \mathbf{b})\mathbf{u} = \Psi \mathbf{f}, \tag{5.4.30}$$

with

$$\mathcal{B}(\varepsilon \mathbf{b}) = \Psi(\partial_t \Psi^t + \sum A_j \partial_j \Psi^t) + \Psi B \Psi^t + \mathbf{1}_- \Psi\, \widetilde{G}(\widetilde{v}_a^\varepsilon, \varepsilon \mathbf{b})\, \Psi^t$$

The matrix $\Psi\mathbb{P}\,\Psi^t$ is constant, and is the matrix of the orthogonal projector of $\mathbb{R}^N$ onto $\mathbb{V}_+$. We will make a first order Taylor expansion around $x_d = 0$ of the matrix $\Psi A_d \Psi^t$. The matrix $\Psi A_d(t,y,0)\Psi^t$ has constant kernel $\mathbb{V}_0$ and constant range $\mathbb{V}_0^\perp$. We denote by Π_0 the (matrix of) the orthogonal projector of $\mathbb{R}^N$ onto $\mathbb{V}_0^\perp$, and $\Pi := \Psi\mathbb{P}\,\Psi^t$. There exists a smooth symmetric matrix $S(t,y)$, uniformly regular (that is $0 < c \leq |\det S| \leq C$ on $\mathbb{R}^d$), such that $[\Pi_0, S] = 0$ and: $\Psi(t,y)A_d(t,y,0)\Psi^t(t,y) = \Pi_0 S \Pi_0 (= S\Pi_0)$. With these notations, the system (5.4.30) takes the form

$$\partial_t \mathbf{u} + \sum_1^d \mathcal{A}_j Z_j \mathbf{u} + \Pi_0 S \Pi_0 \partial_d \mathbf{u} + \frac{1}{\varepsilon}\mathbf{1}_{\{x_d<0\}} \Pi \mathbf{u} + \mathcal{B}(\varepsilon \mathbf{b})\mathbf{u} = \Psi \mathbf{f}. \tag{5.4.31}$$

We note $\mathbf{L}^\varepsilon$ the first order operator defined by the left hand side of (5.4.31). The function $\mathbf{u}$ satisfies also the L^2 estimate (5.4.29) with $\Pi\mathbf{u}$ instead of $\widetilde{\mathbb{P}}\mathbf{v}$. Let us now apply to Equation (5.4.31) the operator

$(\sqrt{\varepsilon}Z)^\alpha = \varepsilon^{\frac{|\alpha|}{2}} Z^\alpha$, where $|\alpha| \le m$. The energy estimate gives

$$\begin{aligned}(5.4.32)\quad &\lambda^{m-|\alpha|+1/2}\|(\sqrt{\varepsilon}Z)^\alpha \mathbf{u}\|_{0,\lambda} + \frac{1}{\varepsilon^{1/2}}\lambda^{m-|\alpha|}\|\Pi(\sqrt{\varepsilon}Z)^\alpha \mathbf{u}_-\|_{0,\lambda,-} \lesssim \\ &\qquad \frac{1}{\lambda^{1/2}}\big(\|f\|_{m,\lambda,\varepsilon} + \lambda^{m-|\alpha|}\|[\mathbf{L}^\varepsilon, (\sqrt{\varepsilon}Z)^\alpha]\mathbf{u}\|_{0,\lambda}\big),\end{aligned}$$

and one is lead to control the commutator in the right hand side. An important point is that Π and Π_0 commute with Z^α. Hence

$$(5.4.33)\qquad \begin{aligned}[\mathbf{L}^\varepsilon, Z^\alpha] &= \sum [\mathcal{A}_j Z_j, Z^\alpha] + \Pi_0[S\partial_d, Z^\alpha]\Pi_0 + [\mathcal{B}(\varepsilon\mathbf{b}), Z^\alpha] \\ &= \sum_{|\beta|\le|\alpha|} \mathcal{B}_\beta Z^\beta + \sum_{|\gamma|\le|\alpha|-1} \mathcal{C}_\gamma Z^\gamma\, \Pi_0\, \partial_d + [\mathcal{B}(\varepsilon\mathbf{b}), Z^\alpha]\end{aligned}$$

where $\mathcal{B}_\beta$ and $\mathcal{C}_\gamma$ are smooth $N \times N$ matrices. Hence

$$(5.4.34)\qquad \begin{aligned}&\lambda^{m-|\alpha|}\|[L^\varepsilon, (\sqrt{\varepsilon}Z)^\alpha]\mathbf{u}\|_{0,\lambda} \lesssim \\ &\|\mathbf{u}\|_{m,\lambda,\varepsilon} + \|\sqrt{\varepsilon}\Pi_0\partial_d\mathbf{u}\|_{m-1,\lambda,\varepsilon} + \|\mathbf{b}\|_{m,\lambda,\varepsilon}|\mathbf{u}|_\infty.\end{aligned}$$

Expressing $\Pi_0\partial_d\mathbf{u}$ by Equation (5.4.31) leads to

$$\begin{aligned}(5.4.35)\quad &\|\sqrt{\varepsilon}\Pi_0\partial_d\mathbf{u}\|_{m-1,\lambda} \lesssim \sqrt{\varepsilon}\|\mathbf{u}\|_{m,\lambda,\varepsilon} + \frac{1}{\sqrt{\varepsilon}}\|\Pi\mathbf{u}_-\|_{m-1,\lambda,-} + \sqrt{\varepsilon}\|\mathbf{f}\|_{m-1,\lambda,\varepsilon} \\ &\qquad + \sqrt{\varepsilon}\,\|\mathbf{b}\|_{m-1,\lambda,\varepsilon}|\mathbf{u}|_\infty.\end{aligned}$$

By replacing in (5.4.34) and (5.4.33) one gets

$$\begin{aligned}(5.4.36)\quad &\lambda^{m-|\alpha|+1/2}\|(\sqrt{\varepsilon}Z)^\alpha \mathbf{u}\|_{0,\lambda} + \frac{1}{\sqrt{\varepsilon}}\lambda^{m-|\alpha|}\|\Pi(\sqrt{\varepsilon}Z)^\alpha \mathbf{u}_-\|_{0,\lambda,-} \lesssim \\ &\frac{1}{\lambda^{1/2}}\big(\|\mathbf{f}\|_{m,\lambda,\varepsilon} + \|\mathbf{u}\|_{m,\lambda,\varepsilon} + \frac{1}{\sqrt{\varepsilon}}\|\Pi\mathbf{u}_-\|_{m,\lambda,\varepsilon,-} + \|\mathbf{b}\|_{m,\lambda,\varepsilon}|\mathbf{u}|_\infty\big).\end{aligned}$$

Summing all inequalities (5.4.36) for $|\alpha| \le m$, and taking λ large enough to absorb in the left hand side the terms $\|\mathbf{u}\|_{m,\lambda,\varepsilon}$ and $\frac{1}{\sqrt{\varepsilon}}\|\Pi\mathbf{u}_-\|_{m,\lambda,\varepsilon,-}$ yields the inequality (5.4.27) for $\mathbf{u}$ and hence for $\mathbf{v}$. $\square$

We need now to estimate the normal derivative of $\mathbf{u}$, the method is classical. We keep the notations of the proof of the previous proposition and work with the unknown $\mathbf{u}$ and Equation (5.4.31). By (5.4.35) we already have an estimate of $\|\sqrt{\varepsilon}\partial_d\Pi_0\mathbf{u}\|_{m-1,\lambda,\varepsilon}$. It remains to estimate $\partial_d(\mathrm{Id}-\Pi_0)\mathbf{u}$. Let us denote $\mathbf{u}_I := (\mathrm{Id}-\Pi_0)\mathbf{u}$, $\mathbf{u}_{II} := \Pi_0\mathbf{u}$ and

$$\mathbb{X} := \sum_{j=0}^{d}(\mathrm{Id}-\Pi_0)\mathcal{A}_j(\mathrm{Id}-\Pi_0)\, Z_j.$$

By applying $(\mathrm{Id}-\Pi_0)$ on the left to the system (5.4.31), since $(\mathrm{Id}-\Pi_0)\Pi = 0$ we get the equation

$$\mathbb{X}\mathbf{u}_I = \sum_{0}^{d} C_j Z_j \mathbf{u}_{II} - (\mathrm{Id}-\Pi_0)\mathcal{B}(\mathbf{b})\mathbf{u} + \mathbf{f}_I$$

and applying the derivation ∂_d leads to an equation of the form

$$\begin{aligned}\mathbb{X}\partial_d\mathbf{u}_I = \sum_{|\alpha|\leq 1} M_\alpha Z^\alpha \mathbf{u} + \sum_{|\beta|\leq 1} N_\beta Z^\beta \partial_n \mathbf{u}_{II} \\ +\partial_d\mathbf{f}_I - \partial_d((\mathrm{Id}-\Pi_0)\mathcal{B}(\mathbf{b})\mathbf{u}).\end{aligned} \tag{5.4.37}$$

The energy estimate for the operator $\mathbb{X}$, applied to Equation (5.4.37)on $\Omega^+_{T_0}$ and on $\Omega^-_{T_0}$ respectively, implies

$$\begin{aligned}\|\partial_d\mathbf{u}_I^\pm\|_{m-2,\lambda,\varepsilon} \lesssim \lambda^{-1}\big(&\|\mathbf{u}\|_{m-1,\lambda,\varepsilon} + \|\partial_d\mathbf{u}_{II}^\pm\|_{m-1,\lambda,\varepsilon} + \|\partial_d\mathbf{f}^\pm\|_{m-2,\lambda,\varepsilon} \\ &+c(R)(\|u\|_{m-2,\lambda,\varepsilon} + \|\mathbf{b}\|_{m-2,\lambda,\varepsilon}|\mathbf{u}|_\infty)\ \big),\end{aligned}$$

and by (5.4.35) we get

$$\begin{aligned}\|\partial_d\mathbf{u}_I^\pm\|_{m-2,\lambda,\varepsilon} \lesssim \frac{c(R)}{\lambda}\ \big(&\|\mathbf{u}\|_{m,\lambda,\varepsilon} + \frac{1}{\varepsilon}\|\Pi\mathbf{u}_-\|_{m-1,\lambda,\varepsilon,-} + \|\mathbf{b}\|_{m-1,\lambda,\varepsilon}|\mathbf{u}|_\infty \\ &+ \|\mathbf{f}\|_{m-1,\lambda,\varepsilon}\ +\ \|\partial_d\mathbf{f}^\pm\|_{m-2,\lambda,\varepsilon}\ \big).\end{aligned} \tag{5.4.38}$$

We now recall an adapted version of Sobolev embeddings. There exist $\kappa > 0$ and $\rho > 0$ such that, if $u \in H^m_{co}(\Omega_{T_0})$ is such that $\partial_d u^\pm \in H^{m-2}_{co}(\Omega^\pm_{T_0})$, the following estimate holds (see [Sue06b], [Guè92]):

$$|u|_\infty \leq \kappa\frac{1}{\varepsilon^\rho}e^{\lambda T}\big(\ \|u\|_{m,\lambda,\varepsilon} + \|\sqrt{\varepsilon}\partial_d u^+\|_{m-2,\lambda,\varepsilon} + \|\sqrt{\varepsilon}\partial_d u^-\|_{m-2,\lambda,\varepsilon}\big) \tag{5.4.39}$$

for all $\lambda > 0$,and all $\varepsilon > 0$. In fact, $\rho = (d+1)/4$, but his has no importance in the proof. Let us recall that k is the order of the approximate solution appearing in the right hand side of (5.4.20). We can now prove that the sequence $\widetilde{w}^{\varepsilon,\nu}$ is bounded, under the assumption that $k-1>\rho$.

Lemma 5.4.3. *Let $\widetilde{w}^{\varepsilon,\nu}$ be the solution of Equation* (5.4.24)*, there exist $\lambda>0$, $a>0$ and $\varepsilon_0>0$ such that:*

$$\|\widetilde{w}^{\varepsilon,\nu}\|_{m,\lambda,\varepsilon} \leq a\varepsilon^{k-1}, \quad |\widetilde{w}^{\varepsilon,\nu}|_\infty \leq 1, \quad \forall \nu \in \mathbb{N}, \forall \varepsilon \in]0,\varepsilon_0]. \tag{5.4.40}$$

Proof. We show the lemma by induction, so we assume that $\widetilde{w}^{\varepsilon,\nu}$ satisfies (5.4.40) (m is fixed). The proposition 5.4.2 gives two constants $C_m(1)$ and $\lambda_m(1)$ associated to the choice $R=1$. Taking first λ large enough in the estimate (5.4.27) we obtain

$$\begin{aligned}\lambda^{1/2}\|w^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon}+&\frac{1}{\varepsilon^{1/2}}\|\Pi w_-^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon,-} \leq \\ &\frac{a}{2}\,\varepsilon^{k-1} + C_m(1)\lambda^{-1/2}a\,\varepsilon^{k-1}\,|w^{\nu+1,\varepsilon}|_\infty.\end{aligned} \tag{5.4.41}$$

The parameter $\lambda>1$ is now fixed. Replacing the estimates (5.4.35) and (5.4.38) in the L^∞ imbedding (5.4.39) gives, for some constant κ_0:

$$\begin{aligned}|w^{\varepsilon,\nu+1}|_\infty \leq \frac{\kappa_0}{\varepsilon^\rho}\big(\|w^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon} + \frac{1}{\varepsilon}\|\Pi w_-^{\varepsilon,\nu+1}\|_{m-1,\lambda,\varepsilon,-} + a\varepsilon^{k-1}|w^{\varepsilon,\nu+1}|_\infty \\ + \varepsilon^{k-1}\big).\end{aligned}$$

and for $\varepsilon>0$ small enough we obtain (for some constant κ_1)

$$|w^{\varepsilon,\nu+1}|_\infty \leq \frac{\kappa_1}{\varepsilon^\rho}\big(\|w^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon} + \frac{1}{\varepsilon}\|\Pi w_-^{\varepsilon,\nu+1}\|_{m-1,\lambda,\varepsilon,-} + \varepsilon^{k-1}\big). \tag{5.4.42}$$

Replacing now (5.4.42) in (5.4.41), and taking again ε small enough so that all the terms $\|w^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon}$ and $\|\Pi w_-^{\varepsilon,\nu+1}\|_{m-1,\lambda,\varepsilon,-}$ can be absorbed in the left hand side (because $k-1>\rho+\frac{1}{2}$), yields the estimate

$$\|w^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon} + \frac{1}{\varepsilon^{1/2}}\|\Pi w_-^{\varepsilon,\nu+1}\|_{m,\lambda,\varepsilon,-} \leq a\,\varepsilon^{k-1} \tag{5.4.43}$$

which implies in particular the first estimate of (5.4.40) at the rank $\nu+1$. We conclude the proof by replacing (5.4.43) in the inequality (5.4.42) which gives, by taking once more ε small enough, the second part of (5.4.40) for $w^{\varepsilon,\nu+1}$. □

Before ending the proof, let us sum up its main steps. By a first change of unknown, we show that the proof of our Theorem can be reduced, without lack of generality, to its proof in the particular case of the penalized problem (5.4.3) (which has boundary conditions well-fitted for a domain penalization approach). So, focusing on the proof of a convergence result, as $\varepsilon \to 0^+$, for the solution $\widetilde{v}^\varepsilon$ of (5.4.3), we first construct an approximate solution then prove the needed energy estimates. Our energy estimates aim at controlling the error $\widetilde{w}^\varepsilon$ between $\widetilde{v}^\varepsilon$ and the constructed approximate solution $\widetilde{v}_a^\varepsilon$. Due to the nonlinearity of the problem, we use a Picard's iterative scheme. We have noted $\widetilde{w}^{\varepsilon,\nu}$ the ν^{th} iterate of the sequence. As usual, we proceed by induction on ν and control, at each step ν, the derivatives of $\widetilde{w}^{\varepsilon,\nu}$ (at step ν, $\widetilde{w}^{\varepsilon,\nu-1}$ is assumed to be known) and the commutators. An important point in the proof is a change of unknown that fixes the kernel, negative space and positive space of the normal coefficient. By projection, elements polarized on the kernel of the normal coefficient (corresponding to the characteristic components) of the new unknown are then treated separately. Lemma 5.4.3 just above allows us to achieve the proof, as it shows that the sequence $w^{\varepsilon,\nu}$ converges in $L^2(\Omega_T)$ towards $w \in H^m_{co}(\Omega_T)$, satisfying the same estimates as $w^{\varepsilon,\nu}$ (it is a classical argument ([Maj84], [Guè90])).

5.5 Proof of Theorem 5.2.6

This section is devoted to the proof of the other main theorem of our paper. Instead of the problem (5.4.3), we will replace the penalization matrix $\widetilde{\mathbb{P}}_+$ by merely Id_N and consider the new simpler problem

$$\left\{ \begin{array}{l} \widetilde{L}\widetilde{v}^\varepsilon + \varepsilon^{-1}\mathbf{1}_{\{x_d<0\}}\widetilde{v}^\varepsilon = \mathbf{1}_{\{x_d>0\}}\,\widetilde{F}(t,x,\widetilde{v}^\varepsilon) \text{ in } \Omega_{T_0}, \\ \widetilde{v}^\varepsilon_{|\Omega_0} = 0. \end{array} \right. \tag{5.5.1}$$

Indeed, by performing once again the change of unknown described in Subsection 5.4.1, it is exactly the problem we obtain. The first step is to find an approximate solution of the problem, and this section is exactly analogous to the preceding section 3.2, but for the new problem (5.5.1). The main difference is that the construction of the approximate solutions shows the existence of boundary layers for this problem. This is not surprising since boundary layers already appeared in the work by J. Droniou devoted to the linear, non characteristic case ([Dro97]), which can be seen as a special case of our result.

We look for an approximate solution of the form

$$v_a^\varepsilon(t,x) = \sum_{j=0}^{M} \varepsilon^j V_j(t,x,x_d/\varepsilon), \tag{5.5.2}$$

where the $V_j(t,x,z)$ for all $j = 0, \cdots, M$ are functions which writes $V_j(t,x,z) = V_j^+(t,x,z)$ if $x_d > 0$ and $z > 0$, and $V_j(t,x,z) = V_j^-(t,x,z)$ if $x_d < 0$ and $z < 0$ with respectively

$$V_j^\pm(t,x,z) = \overline{V}_j^\pm(t,x) + V_j^{*,\pm}(t,y,z) \tag{5.5.3}$$

with $\overline{V}_j^\pm \in H^\infty(\Omega_{T_0}^\pm)$ and $V_j^{*,\pm} \in e^{-\delta_j|z|}H^\infty(\Gamma_{T_0}^\pm \times \mathbb{R}_\pm)$ for some $\delta_j > 0$ depending on $V_j^\pm$. Hence, after substitution of z with x_d/ε the terms in $V_j^{*,\pm}(t,y,x_d/\varepsilon)$ are 'boundary layer terms' which go to 0 in $L^2(\Omega_{T_0}^\pm)$ as $\varepsilon \to 0$, and are exponentially decaying to zero as $|x_d| \to \infty$.

We solve the equation in the half space $x_d > 0$ and in $x_d < 0$ coupled with the transmission condition

$$[(Id - \widetilde{\mathbb{P}}_0)\widetilde{v}^\varepsilon]_{\{x_d=0\}} = 0.$$

which can be also written

$$\widetilde{\mathbb{P}}_-[\widetilde{v}^\varepsilon]_{\{x_d=0\}} = 0, \text{ and } \widetilde{\mathbb{P}}_+[\widetilde{v}^\varepsilon]_{\{x_d=0\}} = 0. \tag{5.5.4}$$

In fact the study of the equations shows that all the $\overline{V}_j$ terms vanish when $x_d < 0$ and all the V_j^* terms vanish when $z > 0$. Hence, in order to simplify the redaction we will directly look for an approximate solution of the form

$$v_a^\varepsilon(t,x) = \sum_{j=0}^{M} \varepsilon^j \overline{V}_j^+(t,x), \text{ on } x_d > 0, \tag{5.5.5}$$

and

$$v_a^\varepsilon(t,x) = \sum_{j=0}^{M} \varepsilon^j V_j^{*,-}(t,y,x_d/\varepsilon), \text{ on } x_d < 0. \tag{5.5.6}$$

The first profile V^0 is now determined by the following three steps.

Step 1. Order ε^{-1}, size < 0. The equation for the terms in ε^{-1} on the side $x_d < 0$ is

$$\widetilde{A_d}(t,y,0)\partial_z V_0^{*,-} + V_0^{*,-} = 0, \tag{5.5.7}$$

which requires the polarization condition

$$V_0^{*,-} \in \mathbb{E}_+(\widetilde{A_d}(t,y,0)) \tag{5.5.8}$$

in order to get the exponential decay as $z \to -\infty$.

Step 2. Order ε^0, size > 0. The equation for the terms of order $O(1)$ on the side $x_d > 0$ is just

$$\widetilde{L}\overline{V}_0^+ = F(\overline{V}_0^+). \tag{5.5.9}$$

Step 3. The boundary condition (5.5.4) at the order ε^0 gives the two conditions (taking (5.5.8) into account):

$$\widetilde{\mathbb{P}}_+ \overline{V}_{0\,|x_d=0}^+ = 0, \tag{5.5.10}$$

and

$$\widetilde{\mathbb{P}}_- \overline{V}_{0\ \ |z=0}^{*,-} = -\widetilde{\mathbb{P}}_- \overline{V}_{0\,|x_d=0}^+. \tag{5.5.11}$$

Now, the system (5.5.9) together with the boundary condition (5.5.10) (and the understood conditions that $\overline{V}_{0\,|t>0}^+ = 0$) is exactly the desired original mixed problem (after the reduction of section 5.4.1), which is well posed, and so $\overline{V}_0^+ = \widetilde{u}$. Then, the second condition (5.5.11) together with the ODE (5.5.7) determines completely $V_0^{*,-}$.

The construction can be continued by induction and all the terms $V_j^{*,-}$, $\overline{V}_j^+$ are determined, for all$j \in \mathbb{N}$. The equation for $V_1^{*,-}$ (in the side $z < 0$) is:

$$\widetilde{A_d}(t,y,0)\partial_z V_1^{*,-} + V_1^{*,-} + LV_0^{*,-} + \partial_d\widetilde{A_d}(t,y,0) z \partial_z V_0^{*,-} = 0, \tag{5.5.12}$$

and the equation for $\overline{V}_1^+$ (in the side $x_d > 0$) is:

$$\widetilde{L}\overline{V}_1^+ = F'(\overline{V}_0^+)\overline{V}_1^+. \tag{5.5.13}$$

More generally for $V_j^{*,-}$, $j \geq 1$

$$\widetilde{A_d}(t,y,0)\partial_z V_j^{*,-} + V_j^{*,-} = q_{j-1}^*, \tag{5.5.14}$$

where the function $q_k^*(t,y,z) \in e^{\delta z} H^\infty(\Gamma_{T_0} \times \mathbb{R}_-)$ depends only of the $V_i^{*,-}$ for $0 \le i \le k$. Equation (5.5.14) can be viewed as an ODE in z, the coordinates (t,y) being parameters. For all (t,y), this equation admits at least a solution in the space $e^{\delta z} H^\infty(\Gamma_{T_0} \times \mathbb{R}_-)$ and two solutions in this space differ from a solution of the homogenous equation (5.5.7). Let us fix a particular solution of the equation $Y_0(z) \in e^{\delta z} H^\infty(\Gamma_{T_0} \times \mathbb{R}_-)$, then all the solutions are of the form

$$V_j^{*,-} = Y_0 + w$$

where w is a solution of (5.5.7) with $w(0) = \widetilde{\mathbb{P}}_+ w(0)$. Consequently, w will be completely determined by its initial value $w(z=0)$. Now the profile equation for the terms of order ε^j is

$$\widetilde{L}\overline{V}_j^+ = \overline{Q}_{j-1}, \tag{5.5.15}$$

where the functions $\overline{Q}_k(t,x) \in H^\infty(\Omega_{T_0}^+)$ depend only on the functions $\overline{V}_i^+$ for $i \le k$. The transmission condition which links $V_j^{*,-}$ and $\overline{V}_j^+$ writes

$$(I - \widetilde{\mathbb{P}}_0) V_j^{*,-}{}_{|z=0} = (I - \widetilde{\mathbb{P}}_0)\overline{V}_j^+{}_{|x_d=0}$$

which splits into

$$\widetilde{\mathbb{P}}_- \overline{V}_j^+{}_{|x_d=0} = \widetilde{\mathbb{P}}_- Y_{0|z=0} \tag{5.5.16}$$

and

$$\widetilde{\mathbb{P}}_+ w_{|z=0} = \widetilde{\mathbb{P}}_+ \overline{V}_j^+{}_{|x_d=0} - \widetilde{\mathbb{P}}_+ Y_{0|z=0}. \tag{5.5.17}$$

As for the step of order 0, the equation (5.5.15) and the boundary condition (5.5.16) determines uniquely $\overline{V}_j^+$. Then, the equation (5.5.14) and the initial condition (5.5.17) determines uniquely w and hence $V_j^{*,-}$.

This shows that one can construct an approximate solution $\widetilde{v}_a^\varepsilon$ of the problem at any given order of approximation, in the sense that $\widetilde{v}_a^\varepsilon$ satisfies now an equation like (5.4.20), but with Id in place of $\widetilde{\mathbb{P}}_+$. Then the estimations of the exact solution and the justification of the asymptotic behavior are proved exactly as in the case of Theorem 5.2.7, in **Subsection 4.3**. For the sake of clarity, we will sketch the proof of this result assuming that the reader is familiar with the proof of our other main Theorem. The goal is to point out the minor changes existing between the two proves.

We have constructed an approximate solution $\widetilde{v}_a^\varepsilon$ of the form (5.4.4) of the problem (5.5.1), satisfying

$$(5.5.18)\quad \begin{cases} \widetilde{L}\widetilde{v}_a^\varepsilon + \varepsilon^{-1}\mathbf{1}_{\{x_d<0\}}\widetilde{v}_a^\varepsilon = \mathbf{1}_{\{x_d>0\}}\,\widetilde{F}(t,x,\widetilde{v}_a^\varepsilon) + \varepsilon^k r^\varepsilon \text{ in } \Omega_{T_0}, \\ \widetilde{v}^\varepsilon_{a|\Omega_0} = 0. \end{cases},$$

where the error term r^ε is piecewise smooth:
(5.5.19)
$$\|r^\varepsilon\|_{L^2(\Omega_{T_0})} = O(1), \quad \|r^\varepsilon_{|\pm x_d>0}\|_{H^m(\Omega^\pm_{T_0})} = O(1), \quad (\varepsilon \to 0^+, \forall m \in \mathbb{N}).$$

We look for an exact solution $\widetilde{v}^\varepsilon$ of the form

$$(5.5.20)\qquad \widetilde{v}^\varepsilon = \widetilde{v}_a^\varepsilon + \varepsilon\widetilde{w}^\varepsilon$$

where $\widetilde{w}^\varepsilon$ is defined by the system

$$\begin{cases} \widetilde{L}\widetilde{w}^\varepsilon + \varepsilon^{-1}\mathbf{1}_{\{x_d<0\}}\widetilde{w}^\varepsilon = \mathbf{1}_{\{x_d>0\}}\,\widetilde{G}(t,x,\widetilde{v}_a^\varepsilon,\varepsilon\widetilde{w}^\varepsilon)\widetilde{w}^\varepsilon + \varepsilon^{k-1}r^\varepsilon \text{ in } \Omega_{T_0}, \\ \widetilde{w}^\varepsilon{}_{|\Omega_0} = 0. \end{cases}$$

where $\widetilde{G}$ is the C^∞ function defined by the Taylor formula:

$$(5.5.21)\qquad \widetilde{G}(t,x,v,\varepsilon w)w = \varepsilon^{-1}\big(\,\widetilde{F}(t,x,v+\varepsilon w) - \widetilde{F}(t,x,v)\,\big).$$

We use a standard Picard's iterative scheme:
(5.5.22)
$$\begin{cases} \widetilde{L}\widetilde{w}^{\varepsilon,\nu+1} + \varepsilon^{-1}\mathbf{1}_-\widetilde{w}^{\varepsilon,\nu+1} - \mathbf{1}_+\,\widetilde{G}(t,x,\widetilde{v}_a^\varepsilon,\varepsilon\widetilde{w}^{\varepsilon,\nu})\widetilde{w}^{\varepsilon,\nu+1} = \varepsilon^{k-1}r^\varepsilon, \\ \widetilde{w}^{\varepsilon,\nu+1}{}_{|\Omega_0} = 0. \end{cases}$$

In order to show the convergence of the sequence $\widetilde{w}^{\varepsilon,\nu}$ we need estimations for the following linear problem

$$(5.5.23)\qquad \begin{cases} \widetilde{L}\mathbf{v} + \varepsilon^{-1}\mathbf{1}_-\mathbf{v} - \mathbf{1}_+\widetilde{G}(\widetilde{v}_a^\varepsilon,\varepsilon\mathbf{b})\mathbf{v} = \varepsilon^{k-1}r^\varepsilon, \\ \mathbf{v}_{|\Omega_0} = 0, \end{cases}$$

where $\widetilde{G}(\widetilde{v}^\varepsilon,\varepsilon\mathbf{b}) = \widetilde{G}(t,x,\widetilde{v}^\varepsilon,\varepsilon\mathbf{b})$ where $\mathbf{b}$ is a given function, which plays the role of $\widetilde{w}^{\varepsilon,\nu}$ when solving the system for the unknown $\mathbf{v} = \widetilde{w}^{\varepsilon,\nu+1}$.

Proposition 5.4.1 still holds for the current problem. We can now prove the following estimate on the linear problem:

Proposition 5.5.1. *Let $R > 0$ and $m \in \mathbb{N}$. There are constants $c_m(R) > 0$ and $\lambda_m(R) > 1$ such that the following holds true. For all $\mathbf{b} \in H^m_{co}(\Omega_{T_0}) \cap L^\infty(\Omega_{T_0})$ such that $|\mathbf{b}|_\infty \le R$, for all $\mathbf{f} \in H^m_{co}(\Omega_{T_0}) \cap H^1_\pm(\Omega_{T_0})$, with $\mathbf{f}_{|t<0} = 0$, the problem (5.5.23) has a unique solution $\mathbf{v} \in H^m_{co}(\Omega_{T_0}) \cap H^1_\pm(\Omega_{T_0})$. Moreover, it follows that $\mathbf{v} \in L^\infty(\Omega_{T_0})$ and the following estimate holds*

(5.5.24)

$$\lambda^{1/2}\|\mathbf{v}\|_{m,\lambda,\varepsilon}+\varepsilon^{-1/2}\,\|\mathbf{v}_-\|_{m,\lambda,\varepsilon,-} \le c_m(R)\,\lambda^{-1/2}\left(\|\mathbf{f}\|_{m,\lambda,\varepsilon}+\|\mathbf{b}\|_{m,\lambda,\varepsilon}|\mathbf{v}|_\infty\right).$$

for all $\lambda \ge \lambda_m(R)$, and all $\varepsilon > 0$.

For the problem at hand, we have the following analogous of Equation (5.4.31):

$$\partial_t \mathbf{u} + \sum_1^d \mathcal{A}_j Z_j \mathbf{u} + \Pi_0 S \Pi_0 \partial_d \mathbf{u} + \frac{1}{\varepsilon}\mathbf{1}_{\{x_d<0\}}\mathbf{u} + \mathcal{B}(\varepsilon\mathbf{b})\mathbf{u} = \Psi\mathbf{f}. \tag{5.5.25}$$

We need now to estimate the normal derivative of $\mathbf{u}$, the method remains classical. We keep the notations of the proof of the previous proposition and work with the unknown $\mathbf{u}$ and Equation (5.5.25). Having proceeded the same way as before, we have already an estimate of $\|\sqrt{\varepsilon}\partial_d\Pi_0\mathbf{u}\|_{m-1,\lambda,\varepsilon}$. The remaining part to estimate is $\partial_d(\mathrm{Id}-\Pi_0)\mathbf{u}$. Denoting $\mathbf{u}_I := (\mathrm{Id}-\Pi_0)\mathbf{u}$, $\mathbf{u}_{II} := \Pi_0\mathbf{u}$ and

$$\mathbb{X} := \sum_{j=0}^d (\mathrm{Id}-\Pi_0)\mathcal{A}_j(\mathrm{Id}-\Pi_0)\,Z_j.$$

By applying $(\mathrm{Id}-\Pi_0)$ on the left to the system (5.5.25), we get the equation

$$\mathbb{X}\mathbf{u}_I + \frac{1}{\varepsilon}(\mathrm{Id}-\Pi_0)\mathbf{u}\mathbf{1}_{x_d<0} = \sum_0^d C_j Z_j \mathbf{u}_{II} - (\mathrm{Id}-\Pi_0)\mathcal{B}(\mathbf{b})\mathbf{u} + \mathbf{f}_I$$

and applying the derivation ∂_d leads to an equation of the form (Π_0 is fixed after our change of unknowns)

(5.5.26)

$$\begin{aligned}\mathbb{X}\partial_d\mathbf{u}_I + \frac{1}{\varepsilon}(\mathrm{Id}-\Pi_0)\partial_d\mathbf{u}\mathbf{1}_{x_d<0} = \sum_{|\alpha|\le 1} M_\alpha Z^\alpha \mathbf{u} + \sum_{|\beta|\le 1} N_\beta Z^\beta \partial_n \mathbf{u}_{II}\\ +\partial_d\mathbf{f}_I - \partial_d((\mathrm{Id}-\Pi_0)\mathcal{B}(\mathbf{b})\mathbf{u}).\end{aligned}$$

The operator $(\mathrm{Id} - \Pi_0)$ is an orthogonal projector thus nonnegative (note well that this argument differs from before), the following estimate holds true:

$$\|\partial_d \mathbf{u}_I^\pm\|_{m-2,\lambda,\varepsilon} \lesssim \lambda^{-1}\big(\|\mathbf{u}\|_{m-1,\lambda,\varepsilon} + \|\partial_d \mathbf{u}_{II}^\pm\|_{m-1,\lambda,\varepsilon} + \|\partial_d \mathbf{f}^\pm\|_{m-2,\lambda,\varepsilon} \\ + c(R)(\|u\|_{m-2,\lambda,\varepsilon} + \|\mathbf{b}\|_{m-2,\lambda,\varepsilon}|\mathbf{u}|_\infty)\ \big),$$

hence
(5.5.27)

$$\|\partial_d \mathbf{u}_I^\pm\|_{m-2,\lambda,\varepsilon} \lesssim \frac{c(R)}{\lambda}\ \big(\|\mathbf{u}\|_{m,\lambda,\varepsilon} + \frac{1}{\varepsilon}\|\mathbf{u}_-\|_{m-1,\lambda,\varepsilon,-} + \|\mathbf{b}\|_{m-1,\lambda,\varepsilon}|\mathbf{u}|_\infty \\ + \|\mathbf{f}\|_{m-1,\lambda,\varepsilon}\ +\ \|\partial_d \mathbf{f}^\pm\|_{m-2,\lambda,\varepsilon}\ \big).$$

By using, like before, an adapted version of Sobolev embeddings, we obtain then:
(5.5.28)

$$|u|_\infty \leq \kappa \frac{1}{\varepsilon^\rho} e^{\lambda T}\big(\ \|u\|_{m,\lambda,\varepsilon} + \|\sqrt{\varepsilon}\partial_d u^+\|_{m-2,\lambda,\varepsilon} + \|\sqrt{\varepsilon}\partial_d u^-\|_{m-2,\lambda,\varepsilon}\big)$$

for all $\lambda > 0$,and all $\varepsilon > 0$. We can now prove that the sequence $\widetilde{w}^{\varepsilon,\nu}$ is bounded, under the assumption that $k - 1 > \rho$.

Lemma 5.5.2. *Let $\widetilde{w}^{\varepsilon,\nu}$ be the solution of Equation* (5.5.22)*, there exist $\lambda > 0$, $a > 0$ and $\varepsilon_0 > 0$ such that:*

$$\text{(5.5.29)} \qquad \|\widetilde{w}^{\varepsilon,\nu}\|_{m,\lambda,\varepsilon} \leq a\varepsilon^{k-1}, \quad |\widetilde{w}^{\varepsilon,\nu}|_\infty \leq 1, \quad \forall \nu \in \mathbb{N}, \forall \varepsilon \in]0, \varepsilon_0].$$

Lemma 5.5.2 just above allows us to conclude the proof, as it shows that the sequence $w^{\varepsilon,\nu}$ converges in $L^2(\Omega_T)$ towards $w \in H^m_{co}(\Omega_T)$, satisfying the same estimates as $w^{\varepsilon,\nu}$.

5.6 Appendix: about hyperbolic systems with discontinuous coefficients.

Let us consider the system (5.2.6) with a fixed $\varepsilon > 0$. The system can be written

$$\text{(5.6.1)} \quad \sum_0^d \partial_j(A_j^\sharp v) + \varepsilon^{-1}\mathbf{1}_{\{x_d<0\}} M v - \sum_{j=0}^{j=d-1} (\partial_j A_j^\sharp) v = \mathbf{1}_{\{x_d<0\}} F(t,x,v)$$

which is meaningful for $u \in L^2(\Omega_T)$. To see that the system is well posed, one shows that it is equivalent to a well posed transmission problem (or initial boundary value problem). Let us note $v^+ = v_{|x_d>0}$ and $v^- = v_{|x_d>0}$, and let us denote by $B(t,y)$ the matrix such that

$$(A_d^\sharp v^+)_{|x_d=+0} - (A_d^\sharp v^-)_{|x_d=-0} =: B \begin{pmatrix} v^- \\ v^+ \end{pmatrix}_{|x_d=0}$$

Lemma 5.6.1. *Let $v \in L^2(\Omega_T)$, v satisfies the system* (5.2.6) *if and only if (v^+, v^-) satisfies the transmission problem*

$$\left\{ \begin{array}{lcl} L^\sharp v^- + \varepsilon^{-1} M v^- & = & 0 \text{ in } \Omega_T^- \\ L^\sharp v^+ & = & F(t,x,v^+) \text{ in } \Omega_T^+ \\ B \begin{pmatrix} v^- \\ v^+ \end{pmatrix}_{|\Gamma_T} = 0, \ v^\pm_{t<0} = 0. & & \end{array} \right. \tag{5.6.2}$$

Proof. If v is solution of (5.6.1), then by restriction on Ω_T^+, v^+ satisfies the corresponding equation in the distributional sense, and that $L^\sharp v^+ \in L^2(\Omega_T^+)$. It follows that the trace $(A_d^\sharp v^+)_{|x_d=0}$ exists in $H^{-1/2}_{loc}(\Gamma_T)$. The same is true for v^- in Ω_T^-. Finally, since there is no Dirac measure in the right hand side of the equation, the transmission condition follows.

Conversely assume that $(v^-, v^+) \in L^2(\Omega_T^-) \times L^2(\Omega_T^+)$ satisfies (5.6.2) and introduce the function $v \in L^2(\Omega_T)$ equal to $v^\pm$ on $\Omega_T^\pm$. The transmission conditions imply that v satisfies the equation (5.6.1) and the lemma is proved. In fact the point is that $A_d^\sharp v$ is *continuous* across $\{x_d = 0\}$:

$$A_d^\sharp v \in C\big(\mathbb{R}_{x_d} : H^{-1/2}_{loc}(\Gamma_T)\big).$$

Hence there is no Dirac measure which appears when applying ∂_d to $A_d^\sharp v$. $\square$

The problem (5.6.2) is an initial boundary value problem with maximally dissipative boundary conditions (see [Ali89],[Sue06a]). Hence, because of this lemma, the results on the non linear mixed hyperbolic problem with characteristic boundary and maximally dissipative boundary conditions can be applied ([Rau85], [Guè90], [Sue06b]), and they show the existence of the solution to the semi-linear problem (5.2.6).

Chapter 6

Pénalisation de problèmes mixtes hyperboliques satisfaisant une condition de Lopatinski Uniforme.

Ce chapitre reprend le papier [For07b] intitulé ”Penalization approach for mixed hyperbolic systems with constant coefficients satisfying a Uniform Lopatinski Condition”, soumis à publication en Avril 2007.

Abstract

In this paper, we describe a new, systematic and explicit way of approximating solutions of mixed hyperbolic systems with constant coefficients satisfying a Uniform Lopatinski Condition via different Penalization approaches.

6.1 Introduction.

In this paper, we describe a new, systematic and explicit way of approximating solutions of mixed hyperbolic systems with constant coefficients satisfying a Uniform Lopatinski Condition via different Penalization approaches. In applied Mathematics like, for instance, in the study of fluids dynamics, the method of penalization is used to treat boundary conditions in the case of complex geometries. By replacing the boundary condition by a singular perturbation of the PDE extended to a larger domain, this method allows the construction of an approximate, often more easily computable, solution. We consider mixed boundary value problems for hyperbolic systems:

$$\partial_t + \sum_{j=1}^{d} A_j \partial_j,$$

on $\{x_d \geq 0\}$, with boundary conditions on $\{x_d = 0\}$. The $n \times n$ real valued matrices A_j are assumed constant. Of course, we assume the coefficients to be constant as a first approach, aiming to generalize the results obtained here in future works. We assume that the boundary $\{x_d = 0\}$ is noncharacteristic, which means that $det A_d \neq 0$. We denote by
$y := (x_1, \dots, x_{d-1})$ and $x := x_d$. The problem writes:

$$\begin{cases} \mathcal{H}u = f, \quad \{x > 0\}, \\ \Gamma u|_{x=0} = \Gamma g, \\ u|_{t<0} = 0 \quad , \end{cases} \tag{6.1.1}$$

where the unknown $u(t,x) \in \mathbb{R}^n$, $\Gamma : \mathbb{R}^n \to \mathbb{R}^p$ is linear and such that $rg\ \Gamma = p$; which implies that Γ can be viewed as a $p \times n$ real valued constant matrix. Let us fix $T > 0$ once and for all for this paper. Let Ω_T^+ denotes the set $[0,T] \times \mathbb{R}_+^d$ and Υ_T denote the set $[0,T] \times \mathbb{R}^{d-1}$. f is a function in $H^k(\Omega_T^+)$, g is a function in $H^k(\Upsilon_T)$, where $k \in \mathbb{N}$ with $k \geq 3$ or $k = \infty$, such that: $f|_{t<0} = 0$ and $g|_{t<0} = 0$. We make moreover the following Hyperbolicity assumption on $\mathcal{H}$:

Assumption 6.1.1. *For all* $(\eta, \xi) \in \mathbb{R}^{d-1} \times \mathbb{R} - \{0\}$, *the eigenvalues of*

$$\sum_{j=1}^{d-1} \eta_j A_j + \xi A_d$$

are real, semi-simple and of constant multiplicity.

Let us introduce now the frequency variable $\zeta := (\gamma, \tau, \eta)$, where $i\tau + \gamma$, with $\gamma \geq 0$, and $\tau \in \mathbb{R}$ stands for the frequency variable dual to t and $\eta = (\eta_1, \ldots, \eta_{d-1})$ where $\eta_j \in \mathbb{R}$ is the frequency variable dual to x_j. We note:

$$A(\zeta) := -(A_d)^{-1}\left((i\tau + \gamma)Id + \sum_{j=1}^{d-1} i\eta_j A_j\right).$$

Denote by M a $N \times N$, complex valued, matrix; $\mathbb{E}_-(M)$[resp $\mathbb{E}_+(M)$] is the linear subspace generated by the generalized eigenvectors associated to the eigenvalues of M with negative [resp positive] real part. If $\mathbb{F}$ and $\mathbb{G}$ denote two linear subspaces of $\mathbb{C}^N$ such that $\dim \mathbb{F} + \dim \mathbb{G} = N$, $det(\mathbb{F}, \mathbb{G})$ denotes the determinant obtained by taking orthonormal bases in each space. Up to the sign, the result is independent of the choice of the bases. We shall now explicit the Uniform Lopatinski Condition assumption:

Assumption 6.1.2. $(\mathcal{H}, \Gamma)$ *satisfies the Uniform Lopatinski Condition i.e for all* ζ *such that* $\gamma > 0$, *there holds:*

$$|det(\mathbb{E}_-(A), \ker \Gamma)| \geq C > 0. \tag{6.1.2}$$

The mixed hyperbolic system (6.1.1) has a unique solution in $H^k(\Omega_T^+)$, and, since $\mathcal{H}$ is hyperbolic with constant multiplicity, for all γ positive, the eigenvalues of A stay away from the imaginary axis. Moreover, as emphasized for instance by Chazarain and Piriou in [CP81] and Métivier in [Mét04], there is a continuous extension of the linear subspace $\mathbb{E}_-(A)$ to $\{\gamma = 0, (\tau, \eta) \neq 0_{\mathbb{R}^d}\}$ that we will denote by $\widetilde{\mathbb{E}}_-(A)$. $\widetilde{\mathbb{E}}_+(A)$ extends as well continuously to $\{\gamma = 0, (\tau, \eta) \neq 0_{\mathbb{R}^d}\}$ and we will denote $\widetilde{\mathbb{E}}_+(A)$ this extension. Moreover, there holds:

$$\widetilde{\mathbb{E}}_-(A) \bigoplus \widetilde{\mathbb{E}}_+(A) = \mathbb{C}^N.$$

We can refer the reader to [CP81], [GMWZ05], [Kre70], or [Mét04] for detailed estimates concerning mixed hyperbolic problems satisfying a Uniform Lopatinski Condition. Moreover, we can refer to [MZ04] for the proof of the continuous extension of the linear subspaces mentioned above in the hyperbolic-parabolic framework.

Remark 6.1.3. *As a consequence of the uniform Lopatinski condition, there holds, for all $\zeta \neq 0$:*

$$rg\ \Gamma = p = \dim \widetilde{\mathbb{E}}_-(A(\zeta)).$$

6.1.1 A Kreiss Symmetrizer Approach.

The main result.

We will now describe a penalization method involving a Kreiss Symmetrizer and a matrix constructed by Rauch in [Rou03], in the construction of our singular perturbation. Note well that we have some freedom in both the choice of the Kreiss Symmetrizer and of Rauch's matrix. Let us denote respectively by $\hat{u}$, $\hat{f}$, and $\hat{g}$ the tangential Fourier-Laplace transform of u, f, and g. Since the Uniform Lopatinski Condition is holding for the mixed hyperbolic system (6.1.1), there is, see [MZ05] a Kreiss symmetrizer S for the problem:

$$\text{(6.1.3)} \qquad \begin{cases} \partial_x \hat{u} = A\hat{u} + \hat{f}, \quad \{x > 0\}, \\ \Gamma \hat{u}|_{x=0} = \Gamma \hat{g}, \end{cases}$$

That is to say there exists a matrix $S(\zeta)$, homogeneous of order zero in ζ, C^∞ in $\mathbb{R}^+ \times \mathbb{R}^d - \{0_{\mathbb{R}^{d+1}}\}$ and there are $\lambda > 0$, $\delta > 0$ and C_1 such that:

- S is hermitian symmetric.
- $\Re\,(SA) \geq \lambda Id$.
- $S \geq \delta Id - C_1 \Gamma^* \Gamma$.

An algebraic result proved by Rauch in [Rou03] can be reformulated as follow:

Lemma 6.1.4. *There is a hermitian symmetric, uniformly definite positive, $N \times N$ matrix B such that:*

$$\ker \Gamma = \mathbb{E}_+((S)^{-1}B).$$

Moreover B depends smoothly of ζ.

Remark 6.1.5. *This result is proved by constructing explicit matrices satisfying the desired properties. Thus, it is not merely an existence result and we can use the explicitly known matrix B in our construction of a penalization operator.*

Let us denote by $R := B^{\frac{1}{2}}$ and $S_R := R^{-1}SR^{-1}$. We will denote by $\mathbb{P}^-$ the projector on $\mathbb{E}_-(S_R)$ parallel to $\mathbb{E}_+(S_R)$ and by $\mathbb{P}^+$ the projector on $\mathbb{E}_+(S_R)$ parallel to $\mathbb{E}_-(S_R)$; $\underline{\mathbb{P}}^-$ and $\underline{\mathbb{P}}^+$ denoting the associated Fourier multiplier. We recall that, denoting by $\mathcal{F}$ the tangential Fourier transform, the Fourier multiplier $\underline{\mathbb{P}}^-(\partial_t, \partial_y, \gamma)$ [resp $\underline{\mathbb{P}}^+(\partial_t, \partial_y, \gamma)$] is then defined, for all $w \in H^k(\mathbb{R}^{d+1})$, and $\gamma > 0$, by:

$$\mathcal{F}\left(\underline{\mathbb{P}}^-(\partial_t, \partial_y, \gamma)w\right) = \mathbb{P}^-(\zeta)\mathcal{F}(w),$$

[resp

$$\mathcal{F}\left(\underline{\mathbb{P}}^+(\partial_t, \partial_y, \gamma)w\right) = \mathbb{P}^+(\zeta)\mathcal{F}(w)],$$

in the future we will rather write:

$$\mathcal{F}\left(\underline{\mathbb{P}}^\pm(\partial_t, \partial_y, \gamma)w\right) = \mathbb{P}^\pm(\zeta)\mathcal{F}(w).$$

We fix, once and for all, $\gamma > 0$ big enough. Let us consider then the solution $\underline{u}^\varepsilon$ of the well-posed Cauchy problem on the whole space (6.1.4):

$$\begin{cases} \mathcal{H}\underline{u}^\varepsilon + \dfrac{1}{\varepsilon}\mathbb{M}\underline{u}^\varepsilon \mathbf{1}_{x<0} = f\mathbf{1}_{x>0} + \dfrac{1}{\varepsilon}\theta\mathbf{1}_{x<0}, \quad \{x \in \mathbb{R}\}, \\ \underline{u}^\varepsilon|_{t<0} = 0, \end{cases} \tag{6.1.4}$$

where

$$\mathbb{M} := -e^{\gamma t} A_d \underline{S}^{-1} \underline{R} \underline{\mathbb{P}}^- \underline{R} e^{-\gamma t},$$
$$\theta := -e^{\gamma t} A_d \underline{S}^{-1} \underline{R} \underline{\mathbb{P}}^- \underline{\Gamma} \widetilde{g},$$

and $\underline{S}(\partial_t, \partial_y)$ [resp $\underline{R}(\partial_t, \partial_y)$] denotes the Fourier multiplier associated to $S(\zeta)$ [resp $R(\zeta)$]. Let us define $\widetilde{g}$ by:

$$\widetilde{g} := e^{-x^2} g.$$

In what follows, $\hat{g}$ will denote the Fourier-Laplace transform of $\widetilde{g}$. Let us denote by

$$\widetilde{\underline{u}} := \underline{u}^- \mathbf{1}_{x<0} + u\mathbf{1}_{x\geq 0} = \underline{u}^- \mathbf{1}_{x\leq 0} + u\mathbf{1}_{x>0}.$$

u denotes the solution of (6.1.1), and thus belongs to $H^k(\Omega_T^+)$. $\underline{u}^-$ is a function belonging to $H^k(\Omega_T^-)$ and such that $\underline{u}^-|_{x=0} = u|_{x=0}$. More precisely, $\underline{u}^-$ can be computed by: $e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1}(\hat{\underline{v}}^- + \mathbb{P}^-\Gamma\hat{g})\right)$, where $\hat{\underline{v}}^-$ is the solution of the problem:

$$\begin{cases} S_R\partial_x\hat{\underline{v}}^- - \mathbb{P}^+S_RA_R\hat{\underline{v}}^- = \mathbb{P}^+S_RA_R\mathbb{P}^-\Gamma\hat{g}, \quad \{x<0\}, \\ \hat{\underline{v}}^-|_{x=0} = \mathbb{P}^+R\hat{u}|_{x=0}, \end{cases}$$

and $\hat{u}$ denotes the Fourier-Laplace transform of the solution u of (6.1.1).

Theorem 6.1.6. *For all $k \in \mathbb{N}$, if $f \in H^k(\Omega_T^+)$ and $g \in H^k(\Upsilon_T)$, then there holds:*

$$\|\underline{u}^\varepsilon - \underline{u}^-\|_{H^{k-3}(\Omega_T^-)} + \|\underline{u}^\varepsilon - u\|_{H^{k-3}(\Omega_T^+)} = \mathcal{O}(\varepsilon),$$

where $\underline{u}^\varepsilon$ denotes the solution of the Cauchy problem (6.1.4) *and u denotes the solution of the mixed hyperbolic problem* (6.1.1)*. If $g=0$ then:*

$$\|\underline{u}^\varepsilon - \underline{u}^-\|_{H^{k-\frac{3}{2}}(\Omega_T^-)} + \|\underline{u}^\varepsilon - u\|_{H^{k-\frac{3}{2}}(\Omega_T^+)} = \mathcal{O}(\varepsilon).$$

Of course, since $\underline{u}^\varepsilon$ is defined for all $\{x \in \mathbb{R}\}$, its limit as $\varepsilon \to 0^+$, $\widetilde{\underline{u}}$ is can be viewed as an 'extension' of u on the fictive domain $\{x < 0\}$. The 'extension' resulting from our method of penalization gives a continuous $\widetilde{\underline{u}}$ across $\{x=0\}$, while the method used in [BR82] gave simply: $\widetilde{\underline{u}}|_{x<0} = 0$. We have the following Corollary:

Corollary 6.1.7. *If f belongs to $L^2(\Omega_T^+)$ and $g=0$ then:*

$$\lim_{\varepsilon\to 0^+} \|\underline{u}^\varepsilon - \widetilde{\underline{u}}\|_{L^2(\Omega_T)} = 0.$$

Moreover there holds:

Corollary 6.1.8. *Assume for example that $f \in H^\infty(\Omega_T^+)$ and $g \in H^\infty(\Upsilon_T)$ then*

$$\|\underline{u}^\varepsilon - u\|_{H^s(\Omega_T^+)} = \mathcal{O}(\varepsilon); \quad \forall s > 0.$$

Remark 6.1.9. *The restriction of the source term of the Cauchy problem* (6.1.4) *to $\{x<0\}$ can be chosen with a lot of freedom.*

One of the interest of this first approach lies in the rate of convergence of $\underline{u}^\varepsilon$ towards u. Indeed, in general, a boundary layer will form near the boundary in this kind of singular perturbation problem. For example in the paper by Bardos and Rauch [BR82], as confirmed by Droniou [Dro97], a boundary layer forms. It is also the case in [PCLS05], as analyzed in our Appendix. There are also boundary layers phenomena in the parabolic context: see the approach proposed by Angot, Bruneau and Fabrie [ABF99] for instance. However, surprisingly, and like in the penalization method proposed by Fornet and Guès in [FG07], our method allows the convergence to occur without formation of any boundary layer on the boundary. As a result, this leads to the kind of sharp stability estimate given in Theorem 6.1.6.

A complementary result.

If we assume that $f \in L^2(\Omega_T^+)$ and $g \in L^2(\Upsilon_T)$, for now, we do not know whether $\underline{u}^\varepsilon$ converges towards $\widetilde{u}$ in $L^2(\Omega_T)$. Approximating the source term and Cauchy data by smooth functions, we propose here another way of penalization that conducts to a Theorem of convergence in $L^2(\Omega_T)$. By the same process, we can prove the following Proposition:

Proposition 6.1.10. *If $f \in H^1(\Omega_T^+)$ and $g \in H^1(\Upsilon_T)$, then*

$$\lim_{\varepsilon \to 0^+} \|\underline{u}^\varepsilon - u\|_{L^2(\Omega_T)} = 0.$$

Let us precise that the method of penalization we introduce now is the same as the previously exposed one in the case where $g = 0$. We define $\widetilde{g} \in L^2(\Omega_T)$, as follow:

$$\widetilde{g}(t, y, x) := e^{-x^2} g(t, y).$$

For all $0 < \varepsilon < \varepsilon_0$, there are f_ε in $H^\infty(\Omega_T^+)$, g_ε in $H^\infty(\Upsilon_T)$, $\widetilde{g}_\varepsilon := e^{-x^2} g_\varepsilon$, and a continuous function ν of ε such that:

$$\lim_{\varepsilon \to 0^+} \nu(\varepsilon) = 0,$$

$$\|f_\varepsilon - f\|_{L^2(\Omega_T^+)} \leq \nu(\varepsilon),$$

and

$$\|\widetilde{g}_\varepsilon^+ - \widetilde{g}\|_{L^2(\Omega_T^+)} + \|\widetilde{g}_\varepsilon^- - \widetilde{g}\|_{L^2(\Omega_T^-)} + \|g_\varepsilon - g\|_{L^2(\Upsilon_T)} \leq \nu(\varepsilon).$$

We denote by $\underline{u}^\varepsilon := (\underline{v}^{\varepsilon+} + \widetilde{g}_\varepsilon^+)\mathbf{1}_{x\geq 0} + (\underline{v}^{\varepsilon-} + \widetilde{g}_\varepsilon^-)\mathbf{1}_{x<0}$, where

$$\underline{v}^\varepsilon = \underline{v}^{\varepsilon+}\mathbf{1}_{x\geq 0} + \underline{v}^{\varepsilon-}\mathbf{1}_{x<0},$$

is defined as the solution of the Cauchy problem:

(6.1.5)

$$\begin{cases} \mathcal{H}\underline{v}^\varepsilon + \dfrac{1}{\varepsilon}\mathbb{M}\underline{v}^\varepsilon\mathbf{1}_{x<0} = (f_\varepsilon - \mathcal{H}\widetilde{g}_\varepsilon^+)\mathbf{1}_{x>0} - \mathcal{H}\widetilde{g}_\varepsilon^-\mathbf{1}_{x<0}, \quad \{x \in \mathbb{R}\}, \\ \underline{v}^\varepsilon|_{t<0} = 0. \end{cases}$$

Theorem 6.1.11. *For some ν, there is a function $\widetilde{u}$, continuous across $\{x = 0\}$, and satisfying $\widetilde{u}|_{x\geq 0} = u$, such that:*

$$\lim_{\varepsilon \to 0^+} \|\underline{u}^\varepsilon - \widetilde{u}\|_{L^2(\Omega_T)} = 0.$$

There also holds:

Theorem 6.1.12. *If $f \in H^1(\Omega_T^+)$ and $g \in H^1(\Upsilon_T)$, then, for some ν, there is $\widetilde{u}$, continuous across $\{x = 0\}$, and satisfying $\widetilde{u}|_{x\geq 0} = u$, such that:*

$$\lim_{\varepsilon \to 0^+} \|\underline{u}^\varepsilon - \widetilde{u}\|_{H^1(\Omega_T)} = 0.$$

6.1.2 A second Approach.

In the first approach we have just introduced, it is necessary to compute a Kreiss's Symmetrizer and a Rauch's matrix. In view of future numerical applications, we will now introduce another method preventing the computation of these matrices. The price to pay is that we need the preliminary computation of v, which is by definition the solution of the Cauchy problem on the free space:

$$\begin{cases} \mathcal{H}v = f, \quad (t, y, x) \in \Omega_T, \\ v|_{t<0} = 0 \quad \forall (y, x) \in \mathbb{R}^d. \end{cases} \tag{6.1.6}$$

The main result.

Let us denote $\mathbf{P}^-(\zeta)$ the spectral projector on $\widetilde{\mathbb{E}}_-(A(\zeta))$ parallel to $\widetilde{\mathbb{E}}_+(A(\zeta))$, and $\mathbf{P}^+(\zeta)$ the spectral projector on $\widetilde{\mathbb{E}}_+(A(\zeta))$ parallel to $\widetilde{\mathbb{E}}_-(A(\zeta))$. Let us introduce $\underline{\mathbf{P}}^\pm(\partial_t, \partial_y, \gamma)$, the Fourier multiplier associated to $\mathbf{P}^\pm(\zeta)$. Let us denote by $\mathbf{\Pi}$ the projector on $\widetilde{\mathbb{E}}_-(A(\zeta))$ parallel to KerΓ, which has a sense because of the Uniform Lopatinski Condition and denote $\underline{\mathbf{\Pi}}$ the associated Fourier multiplier. We define then $\widetilde{h}$ by:

$$\widetilde{h} := e^{-x^2} \left(\underline{\mathbf{P}}^- (e^{-\gamma t} v|_{x=0}) + \underline{\mathbf{\Pi}} e^{-\gamma t} (g - v|_{x=0}) \right),$$

where g denotes the function involved in the boundary condition of the mixed hyperbolic problem (6.1.1). Now, let us consider the following singularly perturbed Cauchy problem on the whole space:

$$\begin{cases} \mathcal{H} u^\varepsilon + \dfrac{1}{\varepsilon} A_d e^{\gamma t} \underline{\mathbf{P}}^- e^{-\gamma t} u^\varepsilon \mathbf{1}_{x<0} = f \mathbf{1}_{x>0} + \dfrac{1}{\varepsilon} A_d e^{\gamma t} \widetilde{h} \mathbf{1}_{x<0}, \\ u^\varepsilon|_{t<0} = 0 \quad . \end{cases} \tag{6.1.7}$$

Let us denote by

$$\widetilde{u} := u^- \mathbf{1}_{x<0} + u \mathbf{1}_{x \geq 0} = u^- \mathbf{1}_{x \leq 0} + u \mathbf{1}_{x>0}.$$

u denotes the solution of (6.1.1) thus belonging to $H^k(\Omega_T^+)$ and u^- is a function belonging to $H^k(\Omega_T^-)$ and such that $u^-|_{x=0} = u|_{x=0}$. More precisely, u^- can be computed by: $e^{\gamma t} \mathcal{F}^{-1}(\mathcal{F}(\widetilde{h}) + \hat{v}^-)$, where $\hat{v}^-$ is the solution of the problem:

$$\begin{cases} \partial_x (\mathbf{P}^+ \hat{v}^-) - A(\mathbf{P}^+ \hat{v}^-) = 0, \quad \{x < 0\}, \\ \mathbf{P}^+ \hat{v}^-|_{x=0} = \mathbf{P}^+ \hat{u}|_{x=0}. \end{cases}$$

and $\hat{u}$ denotes the Fourier-Laplace transform of the solution u of (6.1.1). The problem (6.1.7) is well-posed and, for all $\varepsilon > 0$, there exists a unique
$u^\varepsilon \in H^k(\Omega_T)$ solution. We will fix γ adequately big beforehand. We observe then the following result:

Theorem 6.1.13. *For all $k \in \mathbb{N}$, if $f \in H^k(\Omega_T^+)$ and $g \in H^k(\Upsilon_T)$, then there holds:*

$$\|u^\varepsilon - u^-\|^2_{H^{k-3}(\Omega_T^-)} + \|u^\varepsilon - u\|^2_{H^{k-3}(\Omega_T^+)} = \mathcal{O}(\varepsilon^2),$$

where u^ε denotes the solution of the Cauchy problem (6.1.7) *and u denotes the solution of the mixed hyperbolic problem* (6.1.1).

The singular perturbation involved in the definition of u^ε does not depend either of Kreiss's Symmetrizer or Rauch's matrix. As a result, for this method of penalization far less computations are necessary in order to obtain our singular perturbation. Note well that the proof of the energy estimates in Theorem 6.1.13 is completely different from the proof of the energy estimates in Theorem 6.1.6. Indeed, for our first approach our singularly perturbed problem was treated as a Cauchy problem, contrary to our second approach where it was interpreted as a transmission problem.

Corollary 6.1.14. *Assume for example that* $f \in H^{\infty}(\Omega_T^+)$ *and* $g \in H^{\infty}(\Upsilon_T)$ *then*

$$\|u^{\varepsilon} - u\|_{H^s(\Omega_T^+)} = \mathcal{O}(\varepsilon); \quad \forall s > 0.$$

Remark 6.1.15. *In the case where* $f = 0$, *then the solution* v *of* (6.1.6) *is* $v = 0$ *and thus, the perturbed cauchy problem* (6.1.7) *rewrites:*

$$\begin{cases} \mathcal{H}u^{\varepsilon} + \dfrac{1}{\varepsilon} A_d e^{\gamma t} \underline{\mathbf{P}}^- e^{-\gamma t} u^{\varepsilon} \boldsymbol{1}_{x<0} = \dfrac{1}{\varepsilon} A_d e^{\gamma t} e^{-x^2} \left(\underline{\mathbf{\Pi}} e^{-\gamma t} g \right) \boldsymbol{1}_{x<0}, \quad \{x \in \mathbb{R}\}, \\ u^{\varepsilon}|_{t<0} = 0 \quad . \end{cases}$$

A complementary result.

For all $0 < \varepsilon < \varepsilon_0$, there are f_ε in $H^{\infty}(\Omega_T^+)$, h_ε in $H^{\infty}(\Upsilon_T)$, $\widetilde{h}_\varepsilon := e^{-x^2} h_\varepsilon$, and a continuous function ν of ε such that:

$$\lim_{\varepsilon \to 0^+} \nu(\varepsilon) = 0,$$

$$\|f_\varepsilon - f\|_{L^2(\Omega_T^+)} \leq \nu(\varepsilon),$$

and

$$\|\widetilde{h}_\varepsilon^+ - \widetilde{h}\|_{L^2(\Omega_T^+)} + \|\widetilde{h}_\varepsilon^- - \widetilde{h}\|_{L^2(\Omega_T^-)} + \|h_\varepsilon - h\|_{L^2(\Upsilon_T)} \leq \nu(\varepsilon).$$

There is m_ε such that

$$\underline{\mathbf{P}}^-(\partial_t, \partial_y, \gamma) e^{-\gamma t} m_\varepsilon = h_\varepsilon.$$

We can take for instance:

$$m_\varepsilon = e^{\gamma t} \mathcal{F}^{-1} \left[\mathbf{P}^- \mathcal{F} \left(e^{-\gamma t} v_\varepsilon|_{x=0} \right) + \mathbf{\Pi} \mathcal{F} \left(e^{-\gamma t} (g_\varepsilon - v_\varepsilon|_{x=0}) \right) \right]$$

Where v_ε is the solution of the Cauchy problem:

$$\begin{cases} \mathcal{H}v_\varepsilon = f_\varepsilon, \quad (t,y,x) \in \Omega_T, \\ v_\varepsilon|_{t<0} = 0 \quad . \end{cases}$$

and g_ε belongs to $H^\infty(\Upsilon_T)$ and is such that:

$$\lim_{\varepsilon \to 0} \|g_\varepsilon - g\|_{H^k(\Upsilon_T)} = 0.$$

We define then $\widetilde{m}_\varepsilon$, for all $(t,y,x) \in \Omega_T$ by:

$$\widetilde{m}_\varepsilon = m_\varepsilon e^{-x^2}.$$

The restrictions of $\widetilde{m}_\varepsilon$ to $\pm x > 0$ will be denoted by $\widetilde{m}_\varepsilon^\pm$. Now consider $u^\varepsilon := (\omega^{\varepsilon +} + \widetilde{m}_\varepsilon^+)\mathbf{1}_{x \geq 0} + (\omega^{\varepsilon -} + \widetilde{m}_\varepsilon^-)\mathbf{1}_{x<0}$, where

$$\omega^\varepsilon = \omega^{\varepsilon +}\mathbf{1}_{x \geq 0} + \omega^{\varepsilon -}\mathbf{1}_{x<0}$$

is defined as the solution of the Cauchy problem:

$$\begin{cases} \mathcal{H}\omega^\varepsilon + \dfrac{1}{\varepsilon} A_d e^{\gamma t} \underline{\mathbf{P}}^- e^{-\gamma t} \omega^\varepsilon \mathbf{1}_{x<0} = (f_\varepsilon - \mathcal{H}\widetilde{m}_\varepsilon^+)\mathbf{1}_{x>0} - \mathcal{H}\widetilde{m}_\varepsilon^- \mathbf{1}_{x<0}, \\ \omega^\varepsilon|_{t<0} = 0 \quad . \end{cases}$$

Theorem 6.1.16. *For some ν, there is a function $\widetilde{u}$, continuous across $\{x = 0\}$, and satisfying $\widetilde{u}|_{x \geq 0} = u$, such that:*

$$\lim_{\varepsilon \to 0^+} \|u^\varepsilon - \widetilde{u}\|_{L^2(\Omega_T)} = 0.$$

Of course, there also holds:

Theorem 6.1.17. *If $f \in H^1(\Omega_T^+)$ and $g \in H^1(\Upsilon_T)$, then, for some ν, there is $\widetilde{u}$, continuous across $\{x = 0\}$, and satisfying $\widetilde{u}|_{x \geq 0} = u$, such that:*

$$\lim_{\varepsilon \to 0^+} \|u^\varepsilon - \widetilde{u}\|_{H^1(\Omega_T)} = 0.$$

6.2 Underlying approach leading to the proof of Theorem 6.1.6.

6.2.1 Some preliminaries.

Since the Uniform Lopatinski Condition holds, there is S, homogeneous of order zero in ζ, and such that there are $\lambda > 0$, $\delta > 0$ and C_1 and there holds:

- S is hermitian symmetric.
- $\Re(SA) \geq \lambda Id.$
- $S \geq \delta Id - C_1 \Gamma^* \Gamma.$

S is then called a Kreiss Symmetrizer for the problem:

$$\begin{cases} \partial_x \hat{u} = A\hat{u} + \hat{f}, & \{x > 0\}, \\ \Gamma \hat{u}|_{x=0} = \Gamma \hat{g}, \end{cases} \tag{6.2.1}$$

where $\hat{f}$ and $\hat{g}$ denotes respectively the Fourier-Laplace transforms of f and $\widetilde{g}$; and $\hat{u}$ denotes the Fourier-Laplace transform of the solution u of the well-posed mixed hyperbolic problem (6.1.1). $\hat{u}$ is also solution, for all fixed $\zeta \neq 0$ of the following equation:

$$\begin{cases} S\partial_x \hat{u} = SA\hat{u} + S(A_d)^{-1}\hat{f}, & \{x > 0\}, \\ \Gamma \hat{u}|_{x=0} = \Gamma \hat{g}, \end{cases} \tag{6.2.2}$$

Remark 6.2.1. *Following our current assumptions,* Γ *is independent of* $\zeta \neq 0$, *however, more general boundary conditions, of the form:*

$$\Gamma(\zeta)\hat{u}|_{x=0} = \Gamma(\zeta)\hat{g},$$

can be treated. It would imply taking as boundary condition for (6.1.1)*:*

$$\Gamma_\gamma u|_{x=0} = \Gamma_\gamma g,$$

with for γ *big enough,*

$$\Gamma_\gamma := \underline{\Gamma}(\partial_t, \partial_y)e^{-\gamma t},$$

where, $\underline{\Gamma}(\partial_t, \partial_y)$ *denotes the Fourier multiplier associated to* $\Gamma(\zeta)$, *that is to say is defined by:*

$$\mathcal{F}(\underline{\Gamma}(\partial_t, \partial_y)u) = \Gamma(\zeta)\mathcal{F}(u).$$

Referring for example to [CP81] and [Kre70], Kreiss has proved that the existence of a Kreiss symmetrizer for the symbolic equation is sufficient to prove the well-posedness of the associated pseudodifferential equation (here (6.1.1)). Indeed, multiplying by $\hat{u}$ and integrating by parts the equation:

$$S\partial_x \hat{u} = SA\hat{u} + S(A_d)^{-1}\hat{f}$$

leads to the desired a priori estimates. For all $\zeta \neq 0$, $S(\zeta)$ is hermitian symmetric and definite positive on $\ker \Gamma$. Let us sum up the properties crucial in the proof of the well-posedness of our problem:

Proposition 6.2.2. *For all* $\zeta = (\tau, \gamma, \eta)$ *such that* $\tau^2 + \gamma^2 + \sum_{j=1}^{d+1} \eta_j^2 = 1$, *there holds:*

- $S(\zeta)$ *is hermitian symmetric.*
- $\Re\,(SA)\,(\zeta) := \frac{1}{2}(SA + (SA)^*)(\zeta)$ *is positive definite.*
- $-S(\zeta)$ *is definite negative on* $\ker \Gamma$ *and* $\ker \Gamma$ *is of same dimension as the number of negative eigenvalues in* $-S(\zeta)$.

Note that, by homogeneity of S, it is equivalent for the properties in Proposition 6.2.2 to hold for $|\zeta| = 1$ or for $|\zeta| > 0$. As a consequence of the first point and third point of Proposition 6.2.2 and thanks to an algebraic result proved by Rauch in [Rou03], there holds:

Lemma 6.2.3. *There is a hermitian, uniformly definite positive,* $N \times N$ *matrix* B *such that:*

$$\ker \Gamma = \mathbb{E}_+(S^{-1}B).$$

Moreover B *depends smoothly of* ζ.

The following chapter contains a proof of Lemma 6.2.3 assorted of a detailed construction of B.

6.2.2 Detailed proof of Lemma 6.2.3: Construction of the matrices B solving Lemma 6.2.3.

As we will emphasize in next chapter, Lemma 6.2.3 is a crucial feature in our first method of Penalization. The aim of this chapter is to give a more complete proof rather than simply recalling Rauch's result and, in the process, to precise how the matrices B solving Lemma 6.2.3 are constructed. For all $\zeta \neq 0$, $S(\zeta)$ is hermitian symmetric, uniformly definite positive on $\widetilde{\mathbb{E}}_+(A(\zeta))$, and uniformly definite negative on $\widetilde{\mathbb{E}}_-(A(\zeta))$; as a consequence, $S(\zeta)$ keeps exactly p positive eigenvalues and $N-p$ negative eigenvalues for all $\zeta \neq 0$. Basically, knowing that S is uniformly definite positive on $\ker \Gamma$; we search to express $\ker \Gamma$ in a way involving S. Consider $q \in \ker \Gamma$, since, for all $\zeta \neq 0$, $\mathbb{E}_-(S(\zeta)) \bigoplus \mathbb{E}_+(S(\zeta)) = \mathbb{C}^N$, we can split q in:

$$q := q^+ + q^-$$

with $q^+ \in \mathbb{E}_+(S(\zeta))$ and $q^- \in \mathbb{E}_-(S(\zeta))$.
Since $\dim ker\Gamma = \dim \mathbb{E}_+(S(\zeta)) = p$, these two linear subspaces are in bijection. Let us give the two main ideas behind this proof: one idea is to detail the bijection between $q \in ker\Gamma$ and $q^+ \in \mathbb{E}_+(S(\zeta))$ as it satisfies some constraints, the other is to come down to the model case where the eigenvalues of S are either 1 or -1. Let us denote:

$$\widetilde{S}^{-1} = \begin{bmatrix} -Id_{N-p} & 0 \\ 0 & Id_p \end{bmatrix},$$

In a first step, we will prove the following result:

Proposition 6.2.4. *If we assume that $\mathbb{V}$ is a linear subspace of $\mathbb{C}^N$ of dimension p, and that there is $C > 0$ such that, for all $q \in \mathbb{V}$, there holds:*

$$\langle \widetilde{S}^{-1} q, q \rangle \geq C \langle q, q \rangle,$$

then the two following equivalent properties hold:

- *There is a hermitian symmetric, positive definite matrix $\underline{\widetilde{R}}$, such that:*

$$[q \in \mathbb{V}] \Leftrightarrow \left[\underline{\widetilde{R}}^{-1} q \in \mathbb{E}_+(\underline{\widetilde{R}} \widetilde{S} \underline{\widetilde{R}}) \right],$$

  which is equivalent to:

$$\mathbb{V} = \mathbb{E}_+(\underline{\widetilde{R}}^2 \widetilde{S}).$$

- *There is a hermitian symmetric, positive matrix $\widetilde{R}$, such that:*

$$[q \in \mathbb{V}] \Leftrightarrow \left[\widetilde{R} q \in \mathbb{E}_+(\widetilde{R} \widetilde{S}^{-1} \widetilde{R}) \right],$$

  which is equivalent to:

$$\mathbb{V} = \mathbb{E}_+(\widetilde{S}^{-1} \widetilde{R}^2).$$

Moreover, we can link the two properties by taking:

$$\widetilde{R}^2 = \widetilde{S}\underline{\widetilde{R}}^2\widetilde{S}.$$

Proof. In this proof, we will show how to construct some matrices $\widetilde{R}$ satisfying the required properties. There is a $(N-p)\times p$ matrix $\aleph$ of rank $N-p$ such that $\|\aleph\| \leq 1$ and:

$$\mathbb{V} = \{q \in \mathbb{C}^N, \quad q^- = \aleph q^+\},$$

where q^+[resp q^-] denotes the projector on $\mathbb{E}_+((\widetilde{S})^{-1})$ [resp $\mathbb{E}_-((\widetilde{S})^{-1})$]parallel to $\mathbb{E}_-((\widetilde{S})^{-1})$ [resp $\mathbb{E}_+((\widetilde{S})^{-1})$]. Indeed, $\dim \mathbb{V} = p = \dim \mathbb{E}_+((\widetilde{S})^{-1})$, and $\mathbb{C}^N = \mathbb{E}_-((\widetilde{S})^{-1}) \bigoplus \mathbb{E}_+((\widetilde{S})^{-1})$. Moreover, there ic $C > 0$ such that, for all $q \in \mathbb{V}$, there holds:

$$\langle(\widetilde{S})^{-1}q, q\rangle = -\langle q^-, q^-\rangle + \langle q^+, q^+\rangle \geq C\langle q, q\rangle.$$

and thus

$$|q^+|^2 - |\aleph q^+|^2 \geq C|q|^2,$$

which implies that $\|\aleph\| < 1$. We will show now that, for $\widetilde{R}$ constructed as follow:

$$\widetilde{R} = \begin{bmatrix} Id_{N-p} & -\aleph \\ -\aleph^* & Id_p \end{bmatrix},$$

there holds:

$$[q \in \mathbb{V}] \Leftrightarrow \left[\widetilde{R}q \in \mathbb{E}_+(\widetilde{R}\widetilde{S}^{-1}\widetilde{R})\right].$$

First,we see that the constructed $\widetilde{R}$ is trivially hermitian symmetric and positive definite since $\|\aleph\| < 1$. First, we have:

$$\widetilde{R}\widetilde{S}^{-1}\widetilde{R} = \begin{bmatrix} -Id_{N-p} + NN^* & 0 \\ 0 & Id_p - N^*N \end{bmatrix},$$

and

$$\widetilde{R}q = \begin{pmatrix} q^- - \aleph q^+ \\ -\aleph^* q^- + q^+ \end{pmatrix}.$$

Thus, since $\|\aleph\| < 1$, there holds:

$$\left[\widetilde{R}q \in \mathbb{E}_+(\widetilde{R}\widetilde{S}^{-1}\widetilde{R})\right] \Leftrightarrow \left[q^- - \aleph q^+ = 0\right] \Leftrightarrow [q \in \mathbb{V}].$$

We will now prove that we have:

$$(\widetilde{R})^{-1}\mathbb{E}_+(\widetilde{R}\widetilde{S}^{-1}\widetilde{R}) = \mathbb{E}_+(\widetilde{S}^{-1}\widetilde{R}^2).$$

Since $\widetilde{R}\widetilde{S}^{-1}\widetilde{R}$ is hermitian symmetric, the linear subspace $\mathbb{E}_+(\widetilde{R}\widetilde{S}^{-1}\widetilde{R})$ is generated by the eigenvectors of $\widetilde{R}\widetilde{S}^{-1}\widetilde{R}$ associated to positive eigenvalues. A basis of $(\widetilde{R})^{-1}\mathbb{E}_+(\widetilde{R}\widetilde{S}^{-1}\widetilde{R})$ is thus given by $((\widetilde{R})^{-1}v_j)_j$ where v_j denotes an eigenvector of $\widetilde{R}\widetilde{S}^{-1}\widetilde{R}$ associated to a positive eigenvalue λ_j. We have:

$$\widetilde{R}\widetilde{S}^{-1}\widetilde{R}v_j = \lambda_j v_j.$$

Let us denote $w_j = (\widetilde{R})^{-1}v_j$, we have then:

$$\widetilde{R}\widetilde{S}^{-1}\widetilde{R}^2 w_j = \lambda_j \widetilde{R} w_j \Leftrightarrow \widetilde{S}^{-1}\widetilde{R}^2 w_j = \lambda_j w_j.$$

As a result, w_j is an eigenvector of $S^{-1}\widetilde{R}^2$ associated to the eigenvalue λ_j hence we obtain that:

$$(\widetilde{R})^{-1}\mathbb{E}_+(\widetilde{R}\widetilde{S}^{-1}\widetilde{R}) = \mathbb{E}_+(\widetilde{S}^{-1}\widetilde{R}^2).$$

We can also prove, the same way, that:

$$\underline{\widetilde{R}}\mathbb{E}_+(\underline{\widetilde{R}}\widetilde{S}\underline{\widetilde{R}}) = \mathbb{E}_+(\underline{\widetilde{R}}^2\widetilde{S}).$$

Now, taking

$$\widetilde{R}^2 = \widetilde{S}\underline{\widetilde{R}}^2\widetilde{S},$$

we can check that:

$$\mathbb{E}_+(\widetilde{S}^{-1}\widetilde{R}^2) = \mathbb{E}_+(\underline{\widetilde{R}}^2\widetilde{S}),$$

which concludes the proof. □

Lemma (6.2.3) is a Corollary of the following Proposition:

Proposition 6.2.5. *If S^{-1} denotes a smooth in $\zeta \neq 0$, matrix-valued function in the space of hermitian symmetric matrices with p positive eigenvalues and $N-p$ negative eigenvalues and $\ker\Gamma$ denotes a linear subspace of dimension p and there is $C > 0$ such that, for all $q \in \ker\Gamma$, there holds:*

$$\langle S^{-1}q, q\rangle \geq C\langle q, q\rangle,$$

then the two following equivalent properties hold:

- *There is a smooth in $\zeta \neq 0$, matrix-valued function $\underline{R}$, in the space of hermitian symmetric, positive matrices such that:*

$$[q \in \mathrm{Ker}\Gamma] \Leftrightarrow \left[\forall \zeta \neq 0, \quad \underline{R}^{-1}(\zeta) q \in \mathbb{E}_+(\underline{R}(\zeta) S(\zeta) \underline{R}(\zeta))\right],$$

which is equivalent to:

$$\forall \zeta \neq 0, \quad \mathrm{Ker}\Gamma = \mathbb{E}_+(\underline{R}^2(\zeta) S(\zeta)).$$

- *There is a smooth in $\zeta \neq 0$, matrix-valued function R, in the space of hermitian symmetric, positive matrices such that:*

$$[q \in \mathrm{Ker}\Gamma] \Leftrightarrow \left[\forall \zeta \neq 0, \quad R(\zeta) q \in \mathbb{E}_+(R(\zeta) S^{-1}(\zeta) R(\zeta))\right],$$

which is equivalent to:

$$\forall \zeta \neq 0, \quad \mathrm{Ker}\Gamma = \mathbb{E}_+(S^{-1}(\zeta) R^2(\zeta)).$$

Moreover, for all $\zeta \neq 0$, these two properties can be linked by taking:

$$(R(\zeta))^2 = S(\zeta)(\underline{R}(\zeta))^2 S(\zeta).$$

Proof. We will show here that Proposition 6.2.5 can be deduced from Proposition 6.2.4. For all $\zeta \neq 0$, $S(\zeta)$ is a hermitian symmetric matrix, moreover S depends smoothly of ζ. As a consequence S^{-1} is also a hermitian symmetric matrix depending smoothly of ζ, and as such, there is a nonsingular matrix V such that:

$$\widetilde{S}^{-1} = V^* \left(S^{-1}\right) V.$$

Let us denote Λ the diagonalized version of S^{-1} with eigenvalues sorted by increasing order, then there is Z depending smoothly of ζ such that, for all $\zeta \neq 0$, we have:

$$Z^*(\zeta) = Z^{-1}(\zeta),$$

and

$$\Lambda(\zeta) = Z^*(\zeta) \left(-S^{-1}\right)(\zeta) Z(\zeta).$$

As a consequence, V depends smoothly of ζ since, for all $\zeta \neq 0$:

$$V(\zeta) = (\underline{\Lambda}(\zeta))^{-\frac{1}{2}} Z(\zeta),$$

where $\underline{\Lambda}$ is the diagonal matrix obtained by taking the absolute value of each eigenvalue of Λ. For the sake of simplicity, let us omit the dependence in ζ. Now, for all $q \in V^{-1} \ker \Gamma$, there is $C > 0$, such that:

$$\langle \widetilde{S}^{-1} q, q \rangle = \langle V^* S^{-1} V q, q \rangle = \langle S^{-1}(Vq), (Vq) \rangle \geq C \langle (Vq), (Vq) \rangle.$$

Moreover V is nonsingular, thus there is $C' > 0$, such that, for all $q \in V^{-1} \ker \Gamma$, there holds:

$$\langle \widetilde{S}^{-1} q, q \rangle \geq C' \langle q, q \rangle.$$

Moreover $\dim V^{-1} \ker \Gamma = p$, using Proposition 6.2.4, for all fixed $\zeta \neq 0$, there is a hermitian symmetric, positive definite matrix $\underline{\widetilde{R}}(\zeta)$, such that:

$$V^{-1}(\zeta) \ker \Gamma = \mathbb{E}_+((\underline{\widetilde{R}}(\zeta))^2 \widetilde{S}(\zeta)) = \underline{\widetilde{R}}(\zeta) \mathbb{E}_+(\underline{\widetilde{R}}(\zeta) \widetilde{S}(\zeta) \underline{\widetilde{R}})(\zeta).$$

We will now prove that we can construct $\underline{\widetilde{R}}$ depending smoothly of ζ. First there is a $(N-p) \times p$ matrix $\aleph$ of rank $N-p$, depending smoothly of ζ, such that fore all $\zeta \neq 0$ $\|\aleph(\zeta)\| \leq 1$ and:

$$V^{-1}(\zeta) \ker \Gamma = \{q \in \mathbb{C}^N, \quad q^- = \aleph(\zeta) q^+\},$$

where q^+[resp q^-] denotes the projector on $\mathbb{E}_+((\widetilde{S})^{-1})$ [resp $\mathbb{E}_-((\widetilde{S})^{-1})$]parallel to $\mathbb{E}_-((\widetilde{S})^{-1})$ [resp $\mathbb{E}_+((\widetilde{S})^{-1})$]. $\underline{\widetilde{R}}$ is given, for all $\zeta \neq 0$, by:

$$\underline{\widetilde{R}}(\zeta) = \sqrt{\widetilde{S}^{-1}(\zeta) \widetilde{R}^2(\zeta) \widetilde{S}^{-1}(\zeta)},$$

with $\widetilde{R}$ given by:

$$\widetilde{R}(\zeta) = \begin{bmatrix} Id_{N-p} & -\aleph(\zeta) \\ -\aleph^*(\zeta) & Id_p \end{bmatrix}.$$

Since $\widetilde{S}^{-1} = V^* \left(S^{-1}\right) V$, there holds: $\widetilde{S} = V^* S V$, and, as a consequence:

$$(V \underline{\widetilde{R}})^{-1} \ker \Gamma = \mathbb{E}_+(\underline{\widetilde{R}} V^* S V \underline{\widetilde{R}}).$$

As $\underline{\widetilde{R}} V^* S V \underline{\widetilde{R}}$ is hermitian symmetric, a basis of the linear subspace $\mathbb{E}_+(\underline{\widetilde{R}} V^* S V \underline{\widetilde{R}})$ is given by the eigenvectors of $\underline{\widetilde{R}} V^* S V \underline{\widetilde{R}}$ associated to positive eigenvalues. This leads us to consider $v_j = (V \underline{\widetilde{R}})^{-1} u_j$ satisfying:

$$\underline{\widetilde{R}} V^* S V \underline{\widetilde{R}} v_j = \lambda_j v_j.$$

We have:

$$\underline{\widetilde{R}}V^*SV\underline{\widetilde{R}}(V\underline{\widetilde{R}})^{-1}u_j = \lambda_j(V\underline{\widetilde{R}})^{-1}u_j.$$

hence:

$$(V\underline{\widetilde{R}})\underline{\widetilde{R}}V^*Su_j = \lambda_j u_j.$$

Since $(V\underline{\widetilde{R}})\underline{\widetilde{R}}V^* = (\underline{\widetilde{R}}V^*)^*(\underline{\widetilde{R}}V^*)$ is hermitian symmetric and positive definite, we can then define its square root. We define $\underline{R}$ by:

$$\underline{R} = \sqrt{(\underline{\widetilde{R}}V^*)^*(\underline{\widetilde{R}}V^*)}.$$

Since both $\underline{\widetilde{R}}$ and V depends smoothly of ζ, so does $\underline{R}$. Moreover, there holds:

$$\underline{R}^2Su_j = \lambda_j u_j,$$

which gives:

$$\ker\Gamma = V\underline{\widetilde{R}}\mathbb{E}_+(\underline{\widetilde{R}}V^*SV\underline{\widetilde{R}}) = \mathbb{E}_+(\underline{R}^2S).$$

We have thus proved there is a smooth in $\zeta \neq 0$, matrix-valued function $\underline{R}$, in the space of hermitian symmetric, positive matrices such that:

$$[q \in \mathrm{Ker}\Gamma] \Leftrightarrow \left[\forall\zeta \neq 0, \quad \underline{R}^{-1}(\zeta)q \in \mathbb{E}_+(\underline{R}(\zeta)S(\zeta)\underline{R}(\zeta))\right],$$

which is equivalent to:

$$\forall\zeta \neq 0, \quad \mathrm{Ker}\Gamma = \mathbb{E}_+(\underline{R}^2(\zeta)S(\zeta)).$$

Now consider R defined, for all $\zeta \neq 0$, by:

$$R(\zeta) = \sqrt{S(\zeta)(\underline{R}(\zeta))^2S(\zeta)},$$

$$R(\zeta) = \sqrt{(\underline{\widetilde{R}}(\zeta)V^*(\zeta)S(\zeta))^*(\underline{\widetilde{R}}(\zeta)V^*(\zeta)S(\zeta))}.$$

$\zeta \mapsto R(\zeta)$ is smooth and, for all ζ, $R(\zeta)$ is a hermitian symmetric, positive definite matrix. Moreover, there holds:

$$[q \in \mathrm{Ker}\Gamma] \Leftrightarrow \left[\forall\zeta \neq 0, \quad R(\zeta)q \in \mathbb{E}_+(R(\zeta)S^{-1}(\zeta)R(\zeta))\right],$$

which is equivalent to:

$$\forall\zeta \neq 0, \quad \mathrm{Ker}\Gamma = \mathbb{E}_+(S^{-1}(\zeta)R^2(\zeta)).$$

Let us detail the computation of $R(\zeta)$.

$$R(\zeta) = \sqrt{S(\zeta)V(\zeta)\underline{\widetilde{R}}^2(\zeta)V^*(\zeta)S(\zeta)}.$$

Moreover

$$\underline{\widetilde{R}}^2(\zeta) = \widetilde{S}^{-1}(\zeta)\widetilde{R}^2(\zeta)\widetilde{S}^{-1}(\zeta),$$

we have thus:

$$R(\zeta) = \sqrt{\left(\widetilde{R}(\zeta)\widetilde{S}^{-1}(\zeta)V^*(\zeta)S(\zeta)\right)^* \left(\widetilde{R}(\zeta)\widetilde{S}^{-1}(\zeta)V^*(\zeta)S(\zeta)\right)},$$

which gives:

$$B(\zeta) = \left(\widetilde{R}(\zeta)\widetilde{S}^{-1}(\zeta)V^*(\zeta)S(\zeta)\right)^* \left(\widetilde{R}(\zeta)\widetilde{S}^{-1}(\zeta)V^*(\zeta)S(\zeta)\right).$$

We recall that $\widetilde{R}$ is given, for all $\zeta \neq 0$, by:

$$\widetilde{R}(\zeta) = \begin{bmatrix} Id_{N-p} & -\aleph(\zeta) \\ -\aleph^*(\zeta) & Id_p \end{bmatrix}.$$

and that for all $\zeta \neq 0$, $V(\zeta)$ is given by:

$$V(\zeta) = (\underline{\Lambda}(\zeta))^{-\frac{1}{2}} Z(\zeta),$$

where

$$\Lambda(\zeta) = Z^*(\zeta)\left(-S^{-1}\right)(\zeta)Z(\zeta)$$

with Λ is a diagonal matrix with real coefficients: $(\lambda_1, \ldots, \lambda_N)$, and $\underline{\Lambda}$ denotes the diagonal matrix with diagonal coefficients $(|\lambda_1|, \ldots, |\lambda_N|)$.

Remark 6.2.6. *In the construction of B the only freedom we have resides in the choice of $\aleph$.*

$\square$

6.2.3 A change of dependent variables.

Let us denote by $R := B^{\frac{1}{2}}$ and $\hat{v} := R\hat{u}$. $\hat{v}$ is hence solution of (6.2.3):

$$\begin{cases} R^{-1}SR^{-1}\partial_x\hat{v} = R^{-1}SAR^{-1}\hat{v} + R^{-1}S(A_d)^{-1}\hat{f}, & \{x > 0\}, \\ \Gamma R^{-1}\hat{v}|_{x=0} = \Gamma\hat{g}, \end{cases} \tag{6.2.3}$$

We will adopt the following notations: $S_R := R^{-1}SR^{-1}$, $A_R := RAR^{-1}$, and $\Gamma_R := \Gamma R^{-1}$. We first observe tat:

$$\ker \Gamma_R = R \ker \Gamma = R\mathbb{E}_+((S)^{-1}R^2).$$

but $S_R^{-1} = RS^{-1}R$ thus

$$\ker \Gamma_R = R\mathbb{E}_+(R^{-1}S_R R) = \mathbb{E}_+(S_R).$$

This is where Lemma 6.2.3 is used in a crucial manner. Let us denote by $\mathbb{P}^-$ the projector on $\mathbb{E}_-(S_R)$ parallel to $\mathbb{E}_+(S_R)$ and by by $\mathbb{P}^+$ the projector on $\mathbb{E}_+(S_R)$ parallel to $\mathbb{E}_-(S_R)$; $\underline{\mathbb{P}}^-$ and $\underline{\mathbb{P}}^+$ denoting the associated Fourier multiplier. Since S_R is hermitian symmetric, $\mathbb{P}^-$ is in fact the orthogonal projector on $\mathbb{E}_-(S_R)$. The problem (6.2.3) can then be written:

$$\begin{cases} S_R\partial_x \hat{v} = S_R A_R \hat{v} + R^{-1}S(A_d)^{-1}\hat{f}, & \{x>0\}, \\ \mathbb{P}^-\hat{v}|_{x=0} = \mathbb{P}^-\Gamma\hat{g}, \end{cases}$$

This problem is well-posed because, as a direct Corollary of Proposition 6.2.2, we have:

Proposition 6.2.7. *For all ζ such that $\tau^2+\gamma^2+|\eta|^2 = 1$, there holds:*

- *$S_R(\zeta)$ is hermitian symmetric.*
- *$\Re\left(S_R A_R\right)(\zeta)$ is positive definite.*
- *$-S_R(\zeta)$ is definite negative on* $\ker \Gamma_R$ *and the dimension of* $\ker\Gamma_R$ *is the same as the number of negative eigenvalues of $-S_R(\zeta)$.*

Proof. For the sake of simplicity, let us omit the dependence in ζ in our notations.

- $S_R := R^{-1}SR^{-1}$, and both S and R are hermitian thus S_R is hermitian.
- $S_R A_R = R^{-1}SAR^{-1}$, thus for all $q \in \mathbb{C}^N$, there holds:

  $$2\langle \Re(S_R A_R)q, q\rangle = \langle S_R A_R q, q\rangle + \langle q, S_R A_R q\rangle = \langle R^{-1}SAR^{-1}q, q\rangle + \langle q, R^{-1}SAR^{-1}q\rangle,$$

  since R^{-1} is hermitian, we have then:

  $$= \langle SAR^{-1}q, R^{-1}q\rangle + \langle R^{-1}q, SAR^{-1}q\rangle = 2\langle \Re(SA)R^{-1}q, R^{-1}q\rangle.$$

  Since $\Re(SA)$ is positive definite and R is invertible, $\Re\left(S_R A_R\right)$ is thus positive definite.

- By construction of R, it satisfies $\ker \Gamma_R = \mathbb{E}_+(S_R)$, with S_R hermitian. As a consequence $-S_R$ is definite negative on $\ker \Gamma_R$ and the dimension of $\ker \Gamma_R$ is the same as the number of negative eigenvalues of $-S_R$.

□ Let us mention that, since R and S remains uniformly bounded in $\zeta \neq 0$, $\hat{f}$ and $R^{-1}S(A_d)^{-1}\hat{f}$ belongs to the same space. In a same spirit as [FG07], this suggests the following singular perturbation of (6.2.3):

$$S_R\partial_x \hat{v}^\varepsilon - \frac{1}{\varepsilon}\mathbb{P}^- \hat{v}^\varepsilon \mathbf{1}_{x<0} = S_R A_R \hat{v}^\varepsilon - \frac{1}{\varepsilon}\mathbb{P}^-\Gamma\hat{g}\mathbf{1}_{x<0} + R^{-1}S(A_d)^{-1}\hat{f}, \quad \{x \in \mathbb{R}\},$$

This is equivalent to perturb (6.2.2) as follow:

$$S\partial_x \hat{u}^\varepsilon - \frac{1}{\varepsilon}R\mathbb{P}^- R\hat{u}^\varepsilon \mathbf{1}_{x<0} = SA\hat{u}^\varepsilon - \frac{1}{\varepsilon}R\mathbb{P}^-\Gamma\hat{g}\mathbf{1}_{x<0} + S(A_d)^{-1}\hat{f}, \quad \{x \in \mathbb{R}\},$$

Finally, this induces the following perturbation for (6.1.1):

$$\text{(6.2.4)} \qquad \begin{cases} \mathcal{H}\underline{u}^\varepsilon + \dfrac{1}{\varepsilon}\mathbb{M}\underline{u}^\varepsilon\mathbf{1}_{x<0} = f\mathbf{1}_{x>0} + \dfrac{1}{\varepsilon}\theta\mathbf{1}_{x<0}, \quad \{x \in \mathbb{R}\}, \\ \underline{u}^\varepsilon|_{t<0} = 0, \end{cases}$$

where

$$\mathbb{M} := -e^{\gamma t}A_d\underline{S}^{-1}\underline{R\mathbb{P}^-}\,\underline{R}e^{-\gamma t},$$
$$\theta = -e^{\gamma t}A_d\underline{S}^{-1}\underline{R\mathbb{P}^-}\,\underline{\Gamma\widetilde{g}},$$

and $\underline{S}(\partial_t, \partial_y)$ [resp $\underline{R}(\partial_t, \partial_y)$] denotes the Fourier multiplier associated to $S(\zeta)$ [resp $R(\zeta)$].

6.3 Proof of Theorem 6.1.6, Theorem 6.1.11 and Theorem 6.1.12.

First, we construct an approximate solution of equation (6.2.4) (which is also equation (6.1.4)), then prove suitable energy estimates that ensures $\underline{u}^\varepsilon$ and its approximate solution both converges towards the same limit as
$\varepsilon \to 0^+$. Finally, we use the stability estimates previously established in order to prove Theorem 6.1.11 and Theorem 6.1.12.

6.3.1 Construction of the approximate solution.

$\underline{u}^\varepsilon$ is the solution of the well-posed Cauchy problem:

$$\begin{cases} \mathcal{H}\underline{u}^\varepsilon + \dfrac{1}{\varepsilon}\mathbb{M}\underline{u}^\varepsilon \mathbf{1}_{x<0} = f\mathbf{1}_{x>0} + \dfrac{1}{\varepsilon}\theta \mathbf{1}_{x<0}, \quad \{x \in \mathbb{R}\}, \\ \underline{u}^\varepsilon|_{t<0} = 0. \end{cases}$$

$\underline{u}^\varepsilon$ is moreover the solution of the well-posed Cauchy problem:

$$\begin{cases} \underline{S}A_d^{-1}\mathcal{H}\underline{u}^\varepsilon + \dfrac{1}{\varepsilon}\underline{S}A_d^{-1}\mathbb{M}\underline{u}^\varepsilon \mathbf{1}_{x<0} = \underline{S}A_d^{-1}f\mathbf{1}_{x>0} + \dfrac{1}{\varepsilon}\underline{S}A_d^{-1}\theta \mathbf{1}_{x<0}, \quad \{x \in \mathbb{R}\}, \\ \underline{u}^\varepsilon|_{t<0} = 0. \end{cases}$$

The associated equation after tangential Fourier-Laplace transform writes
:

$$S\partial_x \hat{\underline{u}}^\varepsilon - \frac{1}{\varepsilon}R\mathbb{P}^- R\hat{\underline{u}}^\varepsilon \mathbf{1}_{x<0} - SA\hat{\underline{u}}^\varepsilon = -\frac{1}{\varepsilon}R\mathbb{P}^-\Gamma\hat{g}\mathbf{1}_{x<0} + S(A_d)^{-1}\hat{f}\mathbf{1}_{x>0}, \quad \{x \in \mathbb{R}\}.$$

or alternatively:

$$\begin{cases} \hat{\underline{u}}^\varepsilon = R^{-1}\hat{v}^\varepsilon \\ S_R\partial_x \hat{v}^\varepsilon + \dfrac{1}{\varepsilon}\mathbb{P}^-\hat{v}^\varepsilon \mathbf{1}_{x<0} = S_R A_R \hat{v}^\varepsilon + \dfrac{1}{\varepsilon}\mathbb{P}^-\Gamma\hat{g}\mathbf{1}_{x<0} + R^{-1}S(A_d)^{-1}\hat{f}, \quad \{x \in \mathbb{R}\}, \end{cases}$$

We will use the following formulation as a transmission problem in our construction of an approximate solution:

$$\begin{cases} S_R\partial_x \hat{v}^{\varepsilon+} = S_R A_R \hat{v}^{\varepsilon+} + R^{-1}S(A_d)^{-1}\hat{f}, \quad \{x > 0\}, \\ S_R\partial_x \hat{v}^{\varepsilon-} + \dfrac{1}{\varepsilon}\mathbb{P}^-\hat{v}^{\varepsilon-} = S_R A_R \hat{v}^{\varepsilon-} + \dfrac{1}{\varepsilon}\mathbb{P}^-\Gamma\hat{g}, \quad \{x < 0\}, \\ \hat{v}^{\varepsilon+}|_{x=0^+} = \hat{v}^{\varepsilon-}|_{x=0^-}. \end{cases}$$

For Ω an open regular subset of $\mathbb{R}^{d+1}$, and $\rho \in \mathbb{N}$, let us introduce the weighted spaces $H^\varrho_\gamma(\Omega)$ defined by:

$$H^\varrho_\gamma(\Omega) = \{\varpi \in e^{\gamma t}L^2(\Omega), \|\varpi\|_{H^\varrho_\gamma(\Omega)} < \infty\};$$

where

$$\|\varpi\|^2_{H^\varrho_\gamma(\Omega)} = \sum_{\alpha, |\alpha| \leq \varrho} \gamma^{\rho-|\alpha|}\|e^{-\gamma t}\partial^\alpha \varpi\|^2_{L^2(\Omega)}.$$

We will construct an approximate solution $\underline{u}^\varepsilon_{app}$ of $\underline{u}^\varepsilon$. $\underline{u}^\varepsilon_{app}$ will be constructed as follow:

$$\underline{u}^\varepsilon_{app} = \underline{u}^{\varepsilon+}_{app}\mathbf{1}_{x>0} + \underline{u}^{\varepsilon-}_{app}\mathbf{1}_{x<0},$$

where $\underline{u}^{\varepsilon\pm}_{app}$ is an approximate solution of $\underline{u}^{\varepsilon\pm}$ satisfying the following ansatz:

$$\underline{u}^{\varepsilon\pm}_{app} = \sum_{j=0}^{M} \underline{U}^\pm_j(\zeta, x)\varepsilon^j,$$

where the profiles $\underline{U}^\pm_j$ belong to $H^{k-\frac{3}{2}j}_\gamma(\Omega^\pm_T)$, where $\Omega^\pm_T$ stands for $[0,T]\times\mathbb{R}^d_\pm$. Denote

$$\hat{v}^\varepsilon_{app} = R\mathcal{F}(e^{-\gamma t}\underline{u}^\varepsilon_{app}) := \hat{v}^{\varepsilon+}_{app}\mathbf{1}_{x>0} + \hat{v}^{\varepsilon-}_{app}\mathbf{1}_{x<0}.$$

$\hat{v}^{\varepsilon\pm}_{app}$ is then an approximate solution of $v^{\varepsilon\pm}$ and is of the form:

$$\hat{v}^{\varepsilon\pm}_{app} = \sum_{j=0}^{M} V^\pm_j(\zeta, x)\varepsilon^j;$$

where

$$V^\pm_j = R\mathcal{F}(e^{-\gamma t}\underline{U}^\pm_j),$$

and conversely

$$\underline{U}^\pm_j = e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1}V^\pm_j\right).$$

The profiles $\underline{U}^\pm_j$ can be constructed inductively at any order. Let us show how the first profiles are constructed: Identifying the terms in ε^{-1} gives:

$$\mathbb{P}^- V^-_0 = \mathbb{P}^-\Gamma\hat{g}.$$

Hence, $\mathbb{P}^+V^-_0$ remains to be computed in order to obtain the profile

$$\underline{U}^-_0 = e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1}V^-_0\right).$$

Identifying the terms in ε^0 gives then that V^+_0 is solution of the well-posed problem:

$$\begin{cases} S_R\partial_x V^+_0 = S_R A_R V^+_0 + R^{-1}S(A_d)^{-1}\hat{f}, \quad \{x>0\}, \\ \mathbb{P}^- V^+_0|_{x=0} = \mathbb{P}^-\Gamma\hat{g}. \end{cases} \tag{6.3.1}$$

The associated profile

$$\underline{U}_0^+ = e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1}V_0^+\right)$$

belongs then to $H_\gamma^k(\Omega_T^+)$. Moreover, the problem (6.3.1) is Kreiss-Symmetrizable and thus the trace of the profile $\underline{U}_0^+$, see [CP81] for instance, satisfies:

$$\underline{U}_0^+|_{x=0} \in H_\gamma^k(\Upsilon_T).$$

Since V_0^+ has just be computed, $V_0^-|_{x=0}$ is given by: $V_0^+|_{x=0} - V_0^-|_{x=0} = 0$ and thus, there holds:

$$\mathbb{P}^- V_0^+|_{x=0} = \mathbb{P}^- V_0^-|_{x=0}.$$

Moreover

$$S_R\partial_x V_0^- - \mathbb{P}^- V_1^- = S_R A_R V_0^-, \quad \{x<0\}.$$

Projecting this equation on $\mathbb{E}_+(S_R)$ collinearly to $\mathbb{E}_-(S_R)$ gives then:

$$S_R\partial_x \mathbb{P}^+ V_0^- - \mathbb{P}^+ S_R A_R V_0^- = 0, \quad \{x<0\},$$

Since

$$\mathbb{P}^+ S_R A_R V_0^- = \mathbb{P}^+ S_R A_R \mathbb{P}^+ V_0^- + \mathbb{P}^+ S_R A_R \mathbb{P}^- \Gamma \hat{g},$$

we have then:

$$S_R\partial_x(\mathbb{P}^+ V_0^-) - \mathbb{P}^+ S_R A_R(\mathbb{P}^+ V_0^-) = \mathbb{P}^+ S_R A_R \mathbb{P}^- \Gamma\hat{g}, \quad \{x<0\},$$

and as a consequence, $\mathbb{P}^+ V_0^-$ is solution of the following problem:
(6.3.2)

$$\begin{cases} S_R\partial_x(\mathbb{P}^+ V_0^-) - \mathbb{P}^+ S_R A_R(\mathbb{P}^+ V_0^-) = \mathbb{P}^+ S_R A_R \mathbb{P}^- \Gamma\hat{g} \quad \{x<0\}, \\ \mathbb{P}^+ V_0^-|_{x=0} = \mathbb{P}^+ V_0^+|_{x=0}. \end{cases}$$

Let us precise how (6.3.2) has to be interpreted: we denote $w = \mathbb{P}^+ V_0^-$. w is then totally polarized on $\mathbb{E}_+(S_R)$, and satisfies the problem:

$$(6.3.3) \qquad \begin{cases} \mathbb{P}^+ w = w \\ S_R\partial_x w - \mathbb{P}^+ S_R A_R w = \mathbb{P}^+ S_R A_R \mathbb{P}^- \Gamma\hat{g} \quad \{x<0\}, \\ w|_{x=0} = \mathbb{P}^+ V_0^+|_{x=0}. \end{cases}$$

As we will see, the problem (6.3.3) is Kreiss-Symmetrizable and thus well-posed. Indeed, for all ζ such that $\tau^2 + \gamma^2 + |\eta|^2 = 1$, we have, omitting the dependencies in ζ in our notations:

- For all $q \in \mathbb{C}^N$, there holds:

$$\langle S_R q, q\rangle = \langle q, S_R q\rangle.$$

- Since $Re(S_R A_R)$ is positive definite and $\mathbb{P}^+$ is an orthogonal projector, there is $C > 0$ such that, for all $q \in \mathbb{E}_+(S_R)$, there holds:

$$\langle \mathbb{P}^+ S_R A_R \mathbb{P}^+ q, q\rangle + \langle q, \mathbb{P}^+ S_R A_R \mathbb{P}^+ q\rangle \geq C\langle q, q\rangle.$$

  Indeed, for all $q \in \mathbb{E}_+(S_R)$, there holds:

$$\langle \mathbb{P}^+ S_R A_R \mathbb{P}^+ q, q\rangle = \langle \mathbb{P}^+ S_R A_R \mathbb{P}^+ q, \mathbb{P}^+ q\rangle = \langle S_R A_R \mathbb{P}^+ q, \mathbb{P}^+ q\rangle.$$

- $-S_R$ is definite negative on $\ker \mathbb{P}^+$ that is to say, that there is $c > 0$ such that, for all $q \in \ker \mathbb{P}^+$, there holds:

$$\langle -S_R q, q\rangle \leq -c\langle q, q\rangle.$$

  Moreover $\ker \mathbb{P}^+$ has the same dimension as the number of negative eigenvalues in $-S_R$.

The profile $\underline{U_0^-}$ can then be computed by:

$$\underline{U_0^-} := e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1}(w + \mathbb{P}^-\Gamma\hat{g})\right)$$

belongs to $H^k_\gamma(\Omega_T^-)$, moreover its trace $\underline{U_0^-}|_{x=0}$ belongs to $H^k_\gamma(\Upsilon_T)$. Consider now the equation:

$$\mathbb{P}^- V_1^- = S_R \partial_x V_0^- - S_R A_R V_0^-, \quad \{x < 0\}.$$

Since $\mathbb{P}^- V_1^-|_{x=0} = \mathbb{P}^- V_1^+|_{x=0}$, V_1^+ is solution of the well-posed problem:

$$\begin{cases} S_R \partial_x V_1^+ = S_R A_R V_1^+, \quad \{x > 0\}, \\ \mathbb{P}^- V_1^+|_{x=0} = S_R \partial_x V_0^-|_{x=0} - S_R A_R V_0^-|_{x=0}. \end{cases}$$

Due to the loss of regularity in the boundary condition, the associated profile

$$\underline{U_1^+} = e^{\gamma t}\mathcal{F}^{-1}\left(R^{-1} V_1^+\right)$$

belongs to $H^{k-\frac{3}{2}}_\gamma(\Omega_T^+)$, moreover its trace $\underline{U_1^+}|_{x=0}$ belongs to $H^{k-\frac{3}{2}}_\gamma(\Upsilon_T)$. Moreover, applying $\mathbb{P}^+$ to the equation:

$$\mathbb{P}^- V_2^- + S_R A_R V_1^- = S_R \partial_x V_1^-, \quad \{x < 0\},$$

we obtain:

$$\begin{cases} S_R \partial_x \mathbb{P}^+ V_1^- = \mathbb{P}^+ S_R A_R \mathbb{P}^+ V_1^- + \mathbb{P}^+ S_R A_R \mathbb{P}^- V_1^-, \quad \{x < 0\}, \\ \mathbb{P}^+ V_1^-|_{x=0} = \mathbb{P}^+ V_1^+|_{x=0}. \end{cases}$$

As before, let us take $\mathbb{P}^+ V_1^-$ as the unknown of the well-posed problem:

$$\begin{cases} S_R \partial_x (\mathbb{P}^+ V_1^-) - \mathbb{P}^+ S_R A_R (\mathbb{P}^+ V_1^-) = \mathbb{P}^+ S_R A_R \left(S_R \partial_x V_0^- - S_R A_R V_0^- \right), \ \{x < 0\}, \\ (\mathbb{P}^+ V_1^-)|_{x=0} = \mathbb{P}^+ V_1^+|_{x=0}. \end{cases}$$

This problem is Kreiss-Symmetrizable since, for all ζ such that $\tau^2 + \gamma^2 + |\eta|^2 = 1$, there holds:

- For all $q \in \mathbb{C}^N$, there holds:

$$\langle S_R q, q \rangle = \langle q, S_R q \rangle.$$

- There is $C > 0$ such that for all $q \in \mathbb{E}_+(S_R)$, there holds:

$$\langle \mathbb{P}^+ S_R A_R \mathbb{P}^+ q, q \rangle + \langle q, \mathbb{P}^+ S_R A_R \mathbb{P}^+ q \rangle \geq C \langle q, q \rangle.$$

- $-S_R$ is definite negative on $\ker \mathbb{P}^+$ that is to say, that there is $c > 0$ such that, for all $q \in \ker \mathbb{P}^+$, there holds:

$$\langle -S_R q, q \rangle \leq -c \langle q, q \rangle.$$

  Moreover $\ker \mathbb{P}^+$ has the same dimension as the number of negative eigenvalues in $-S_R$.

However, due to a loss of regularity in both the source term and the boundary condition, the associated profile

$$\underline{U_1^-} = e^{\gamma t} \mathcal{F}^{-1} \left(R^{-1} \left(\mathbb{P}^+ V_1^- + S_R \partial_x V_0^- - S_R A_R V_0^- \right) \right)$$

belongs to $H_\gamma^{k-\frac{3}{2}}(\Omega_T^-)$. The construction of the following profiles can be pursued at any order the same way. In practice, we take:

$$u_{app}^{\varepsilon +} = \underline{U_0^+},$$

$$u_{app}^{\varepsilon -} = \underline{U_0^-} + \varepsilon \underline{U_1^-}.$$

As a result, the approximate solution writes $\underline{u}^{\varepsilon}_{app} := \underline{u}^{\varepsilon+}_{app}\mathbf{1}_{x>0}+\underline{u}^{\varepsilon-}_{app}\mathbf{1}_{x<0}$; where $\underline{u}^{\varepsilon+}_{app}$ belongs to $H^k_\gamma(\Omega^+_T)$ and $\underline{u}^{\varepsilon-}_{app}$ belongs to $H^{k-\frac{3}{2}}_\gamma(\Omega^-_T)$. $\underline{u}^{\varepsilon}_{app}$ is then solution of a well-posed problem of the form:

$$\begin{cases} \mathcal{H}\underline{u}^{\varepsilon}_{app} + \dfrac{1}{\varepsilon}\mathbb{M}\underline{u}^{\varepsilon}_{app}\mathbf{1}_{x<0} = f\mathbf{1}_{x>0} + \dfrac{1}{\varepsilon}\theta\mathbf{1}_{x<0} + \varepsilon\underline{r}^{\varepsilon}, \quad \{x\in\mathbb{R}\}, \\ \underline{u}^{\varepsilon}_{app}|_{t<0} = 0 \quad . \end{cases} \tag{6.3.4}$$

Where $\underline{r}^{\varepsilon} := \underline{r}^{\varepsilon+}\mathbf{1}_{x>0} + \underline{r}^{\varepsilon-}\mathbf{1}_{x<0}$, with $\underline{r}^{\varepsilon+} \in H^{k-\frac{5}{2}}_\gamma(\Omega^+_T)$ and $\underline{r}^{\varepsilon-} \in H^{k-3}_\gamma(\Omega^-_T)$.

Remark 6.3.1. *In the case where $g=0$, the loss of regularity in the profiles is delayed by one step. As a result, in this case we obtain:*

$$\underline{u}^{\varepsilon+}_{app} \in H^k_\gamma(\Omega^+_T),$$
$$\underline{u}^{\varepsilon-}_{app} \in H^k_\gamma(\Omega^-_T),$$
$$\underline{r}^{\varepsilon+} \in H^k_\gamma(\Omega^+_T),$$
$$\underline{r}^{\varepsilon-} \in H^{k-\frac{3}{2}}_\gamma(\Omega^-_T).$$

6.3.2 Stability estimates

We will begin by proving energy estimates on the following equation:

$$S_R A_R \hat{e}^{\varepsilon} - S_R\partial_x\hat{e}^{\varepsilon} + \frac{1}{\varepsilon}\mathbb{P}^{-}\hat{e}^{\varepsilon}\mathbf{1}_{x<0} = \varepsilon\hat{r}^{\varepsilon}, \quad \{x\in\mathbb{R}\}, \tag{6.3.5}$$

where $\hat{e}^{\varepsilon} = R\left(\mathcal{F}(e^{-\gamma t}\underline{u}^{\varepsilon}) - \mathcal{F}(e^{-\gamma t}\underline{u}^{\varepsilon}_{app})\right) := \hat{w}^{\varepsilon}$; with $w^{\varepsilon} = \underline{u}^{\varepsilon} - \underline{u}^{\varepsilon}_{app}$. Refering to (6.3.4), w^{ε} is the solution of the Cauchy problem:

$$\begin{cases} \mathcal{H}w^{\varepsilon} + \dfrac{1}{\varepsilon}\mathbb{M}w^{\varepsilon}\mathbf{1}_{x<0} = \varepsilon\underline{r}^{\varepsilon}, \\ w^{\varepsilon}|_{t<0} = 0 \quad . \end{cases} \tag{6.3.6}$$

For a fixed positive ε, the perturbation is nonsingular and thus the principal part of the pseudodifferential operator $\mathcal{H} + \frac{1}{\varepsilon}\mathbb{M}$ is the same as the principal part of $\mathcal{H}$. Hence, there is a unique solution of the Cauchy problem (6.3.6): w^{ε} which belongs to $H^{k-3}_\gamma(\Omega_T)$. In order to simplify

the notations, in this chapter we shall denote by L^2 and H^ϱ_γ the spaces: $L^2(\Omega_T)$ and $H^\varrho_\gamma(\Omega_T)$.
We recall the definition of the weighted spaces: $H^\varrho_\gamma(\Omega_T)$ for $\rho \in \mathbb{N}$.

$$H^\varrho_\gamma(\Omega_T) = \{\varpi \in e^{\gamma t} L^2(\Omega_T), \|\varpi\|_{H^\varrho_\gamma(\Omega_T)} < \infty\};$$

where

$$\|\varpi\|^2_{H^\varrho_\gamma(\Omega_T)} = \sum_{\alpha, |\alpha| \leq \varrho} \gamma^{\rho - |\alpha|} \|e^{-\gamma t} \partial^\alpha \varpi\|^2_{L^2(\Omega_T)}.$$

For fixed positive ε, there holds:

$$\int_{-\infty}^{\infty} \partial_x \langle S_R \hat{e}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2} \; dx = 0.$$

$$\Leftrightarrow \int_{-\infty}^{\infty} 2Re\langle S_R \partial_x \hat{e}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2} \; dx = 0.$$

Using the equation, we have then:

$$\int_{-\infty}^{\infty} Re\langle S_R A_R \hat{e}^\varepsilon + \frac{1}{\varepsilon} \mathbb{P}^- \hat{e}^\varepsilon - \varepsilon \hat{r}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2} \; dx = 0.$$

which is equivalent to:

$$\int_{-\infty}^{\infty} Re\langle S_R A_R \hat{e}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2} \; dx = \frac{-1}{\varepsilon} \int_{-\infty}^{\infty} Re\langle \mathbb{P}^- \hat{e}^\varepsilon \varepsilon \hat{r}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2} \; dx$$

$$+\varepsilon \int_{-\infty}^{\infty} Re\langle \hat{r}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2)} \; dx.$$

But $Re\langle S_R A_R \hat{e}^\varepsilon, \hat{e}^\varepsilon \rangle = \langle Re\,(S_R A_R)\, \hat{e}^\varepsilon, \hat{e}^\varepsilon \rangle$ and $Re\,(S_R A_R)$ is positive definite, hence there is $C > 0$, independent of ε such that:

$$C\gamma \|\hat{e}^\varepsilon\|^2_{L^2} + \frac{1}{\varepsilon} \int_{-\infty}^{\infty} Re\langle \mathbb{P}^- \hat{e}^\varepsilon, \hat{e}^\varepsilon \rangle \leq \int_{-\infty}^{\infty} Re\langle \varepsilon \hat{r}^\varepsilon, \hat{e}^\varepsilon \rangle_{L^2} \; dx.$$

Thus, because $\mathbb{P}^-$ is an orthogonal projector, for all positive λ, there holds:

$$C\gamma \|\hat{e}^\varepsilon\|^2_{L^2} + \frac{1}{\varepsilon} \|\mathbb{P}^- \hat{e}^\varepsilon\|^2_{L^2} \leq \frac{1}{2} \left(\frac{\gamma}{\lambda} \|\hat{e}^\varepsilon\|^2_{L^2} + \frac{\lambda}{\gamma} \|\varepsilon \hat{r}^\varepsilon\|^2_{L^2} \right).$$

Choosing λ big enough we have $C - \frac{\varepsilon}{2\lambda} > 0$ and the following energy estimate:

$$\left(C - \frac{\varepsilon}{2\lambda}\right)\gamma\|\hat{e}^\varepsilon\|_{L^2}^2 + \frac{1}{\varepsilon}\|\mathbb{P}^-\hat{e}^\varepsilon\|_{L^2}^2 \leq \frac{\varepsilon^2\lambda}{2\gamma}\|\hat{r}^\varepsilon\|_{L^2}^2.$$

This shows that $\hat{e}^\varepsilon$ converges towards zero in L^2 when ε tends to zero, with a rate in $\mathcal{O}(\varepsilon)$. We recall that $\hat{e}^\varepsilon$ is given by:

$$\hat{e}^\varepsilon := R\mathcal{F}\left(e^{-\gamma t}(\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon)\right),$$

and $\hat{r}^\varepsilon$ is given by:

$$\hat{r}^\varepsilon := R\mathcal{F}\left(e^{-\gamma t}\underline{r}^\varepsilon\right).$$

Moreover, since R and $\mathbb{P}^-$ are two uniformly bounded, uniformly definite positive matrices, there are two positive real numbers α and β such that, for all $\zeta \neq 0$ and $x \in \mathbb{R}$, there holds:

- $\alpha\|\mathcal{F}\left(e^{-\gamma t}(\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon)\right)\|_{L^2}^2 \leq \|\hat{e}^\varepsilon\|_{L^2}^2$.
- $\alpha\|\mathbb{P}^-\mathcal{F}\left(e^{-\gamma t}(\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon)\right)\|_{L^2}^2 \leq \|\mathbb{P}^-\hat{e}^\varepsilon\|_{L^2}^2$.
- $\|\hat{r}^\varepsilon\|_{L^2}^2 \leq \beta\|\mathcal{F}\left(e^{-\gamma t}\underline{r}^\varepsilon\right)\|_{L^2}^2$.

Applying then Plancherel's equality we obtain then:

$$\left(C - \frac{\varepsilon}{2\lambda}\right)\gamma\|\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon\|_{e^{\gamma t}L^2}^2 + \frac{1}{\varepsilon}\|\mathbb{M}\left(\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon\right)\|_{e^{\gamma t}L^2}^2 \leq \frac{\beta}{\alpha}\frac{\varepsilon^2\lambda}{2\gamma}\|\underline{r}^\varepsilon\|_{e^{\gamma t}L^2}^2.$$

We have thus proved there are two positive constants c and C such that:

$$c\gamma\|\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon\|_{e^{\gamma t}L^2}^2 + \frac{1}{\varepsilon}\|\mathbb{M}\left(\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon\right)\|_{e^{\gamma t}L^2}^2 \leq \frac{C\varepsilon^2}{\gamma}\|\underline{r}^\varepsilon\|_{e^{\gamma t}L^2}^2.$$

Let us denote by $\|.\|_{H_\gamma^\varrho}^* := \sqrt{\|.\|_{H_\gamma^\varrho(\Omega_T^+)}^2 + \|.\|_{H_\gamma^\varrho(\Omega_T^-)}^2}$. More generally, when $\underline{r}^\varepsilon \in H^\varrho$, there is two positive constants c_ρ and C_ρ such that:

$$c_\rho\gamma\|\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon\|_{H_\gamma^\varrho}^{*2} + \frac{1}{\varepsilon}\|\mathbb{M}\left(\underline{u}_{app}^\varepsilon - \underline{u}^\varepsilon\right)\|_{H_\gamma^\varrho}^{*2} \leq \varepsilon^2\frac{C_\rho}{\gamma}\|\underline{r}^\varepsilon\|_{H_\gamma^\varrho}^{*2}.$$

As we have seen during the construction of the profiles, $\varrho = k - 3$ in general and $\rho = k - \frac{3}{2}$ in the case where $g = 0$.

6.3.3 End of the proof of Theorem 6.1.6.

As a consequence of our stability estimate, there holds:

$$\|\underline{u}^{\varepsilon}_{app} - \underline{u}^{\varepsilon}\|^2_{H^{k-3}(\Omega_T^-)} + \|\underline{u}^{\varepsilon}_{app} - \underline{u}^{\varepsilon}\|^2_{H^{k-3}(\Omega_T^+)} = \mathcal{O}(\varepsilon^2).$$

Moreover, by construction of $\underline{u}^{\varepsilon}_{app}$, there holds:

$$\|\underline{u}^{\varepsilon}_{app} - \underline{u}^{-}\|^2_{H^{k-3}(\Omega_T^-)} + \|\underline{u}^{\varepsilon}_{app} - u\|^2_{H^{k-3}(\Omega_T^+)} = \mathcal{O}(\varepsilon^2).$$

Hence, we have proved that:

$$\|\underline{u}^{\varepsilon} - \underline{u}^{-}\|^2_{H^{k-3}(\Omega_T^-)} + \|\underline{u}^{\varepsilon} - u\|^2_{H^{k-3}(\Omega_T^+)} = \mathcal{O}(\varepsilon^2).$$

By the same arguments, if $g = 0$, there holds:

$$\|\underline{u}^{\varepsilon} - \underline{u}^{-}\|^2_{H^{k-\frac{3}{2}}(\Omega_T^-)} + \|\underline{u}^{\varepsilon} - u\|^2_{H^{k-\frac{3}{2}}(\Omega_T^+)} = \mathcal{O}(\varepsilon^2).$$

This completes the proof of Theorem 6.1.6.

6.3.4 Proof of Theorem 6.1.11 and Theorem 6.1.12.

We recall that, by Assumption, there holds:

$$f \in H^k(\Omega_T^+),$$

and

$$g \in H^k(\Upsilon_T).$$

We denote then $\widetilde{g}$ the function defined by:

$$\widetilde{g}(t,y,x) := e^{-x^2} g(t,y),$$

and by $\widetilde{g}^{\pm}$ the restrictions of $\widetilde{g}$ to $\pm x > 0$. For all $0 < \nu < 1$, there is f_ν in $H^\infty(\Omega_T^+)$ and $\widetilde{g}_\nu := e^{-x^2} g_\nu$, such that:

$$\|f_\nu - f\|_{H^k(\Omega_T^+)} \leq \nu,$$

and

$$\|\widetilde{g}^+_\nu - \widetilde{g}\|_{H^k(\Omega_T^+)} + \|\widetilde{g}^-_\nu - \widetilde{g}\|_{H^k(\Omega_T^-)} + \|g_\nu - g\|_{H^k(\Upsilon_T)} \leq \nu,$$

with $\widetilde{g}_\nu^{\pm}|_{x=0} = g_\nu$. We denote then by $\underline{u}_\nu$ the solution of the mixed hyperbolic problem:

$$\begin{cases} \mathcal{H}\underline{u}_\nu = f_\nu, \quad \{x>0\}, \\ \Gamma \underline{u}_\nu|_{x=0} = \Gamma g_\nu, \\ \underline{u}_\nu|_{t<0} = 0 \quad . \end{cases}$$

Let us denote $\underline{v}_\nu := \underline{u}_\nu - \widetilde{g}_\nu^{+}$, $\underline{v}_\nu$ is solution of the mixed hyperbolic problem:

$$\begin{cases} \mathcal{H}\underline{v}_\nu = f_\nu - \mathcal{H}\widetilde{g}_\nu^{+}, \quad \{x>0\}, \\ \Gamma \underline{v}_\nu|_{x=0} = 0, \\ \underline{v}_\nu|_{t<0} = 0 \quad . \end{cases}$$

and by $\underline{u}_\nu^\varepsilon := (\underline{v}_\nu^{\varepsilon+} + \widetilde{g}_\nu^{+})\mathbf{1}_{x\geq 0} + (\underline{v}_\nu^{\varepsilon-} + \widetilde{g}_\nu^{-})\mathbf{1}_{x<0}$, where

$$\underline{v}_\nu^\varepsilon = \underline{v}_\nu^{\varepsilon+}\mathbf{1}_{x\geq 0} + \underline{v}_\nu^{\varepsilon-}\mathbf{1}_{x<0}$$

is defined as the solution of the Cauchy problem:

$$\begin{cases} \mathcal{H}\underline{v}_\nu^\varepsilon + \dfrac{1}{\varepsilon}\mathbb{M}\underline{v}_\nu^\varepsilon \mathbf{1}_{x<0} = (f_\nu - \mathcal{H}\widetilde{g}_\nu^{+})\mathbf{1}_{x>0} - \mathcal{H}\widetilde{g}_\nu^{-}\mathbf{1}_{x<0}, \quad \{x \in \mathbb{R}\}, \\ \underline{v}_\nu^\varepsilon|_{t<0} = 0. \end{cases}$$

As a consequence of the result we have just proved, we have then, for all fixed $\nu > 0$, $\forall s > 0$:

$$\|\underline{v}_\nu^\varepsilon - \underline{v}_\nu^{-}\|^2_{H^s(\Omega_T^-)} + \|\underline{v}_\nu^\varepsilon - \underline{v}_\nu\|^2_{H^s(\Omega_T^+)} \leq c_\nu \varepsilon^2,$$

with $\underline{v}_\nu^{-}|_{x=0} = \underline{v}_\nu|_{x=0}$. Let us define: $\underline{u}_\nu^{-} := \underline{v}_\nu^{-} + \widetilde{g}_\nu^{-}\mathbf{1}_{x<0}$, we have thus:

$$\|\underline{u}_\nu^\varepsilon - \underline{u}_\nu^{-}\|^2_{H^s(\Omega_T^-)} + \|\underline{u}_\nu^\varepsilon - u_\nu\|^2_{H^s(\Omega_T^+)} \leq c_\nu \varepsilon^2,$$

with $\underline{u}_\nu^{-}|_{x=0} = u_\nu|_{x=0}$. $\underline{u}_\nu^{-} := \underline{v}_\nu^{-} + \widetilde{g}_\nu^{-}$, with $\underline{v}_\nu^{-} := e^{\gamma t}\mathcal{F}^{-1}(R^{-1}\underline{\hat{w}}_\nu^{-})$ and $\underline{\hat{w}}_\nu^{-}$ is solution of the Kreiss-symmetrizable problem:

$$\begin{cases} S_R\partial_x \underline{\hat{w}}_\nu^{-} - \mathbb{P}^+ S_R A_R \underline{\hat{w}}_\nu^{-} = -R^{-1}S(A_d)^{-1}\mathcal{F}(e^{-\gamma t}\mathcal{H}\widetilde{g}_\nu^{-}), \quad \{x<0\}, \\ \underline{\hat{w}}_\nu^{-}|_{x=0} = \mathbb{P}^+ R\hat{v}_\nu|_{x=0}, \end{cases}$$

where $\hat{v}_\nu$ stands for the Fourier-Laplace transform of v_ν. In this equation, $\underline{\hat{w}}_\nu$ depends from ν only through its boundary condition. Moreover, since v_ν is solution of a mixed hyperbolic problem satisfying a Uniform Lopatinski Condition, there is $C_1 > 0$ such that: $\|\underline{\hat{v}}_\nu - \underline{\hat{v}}\|_{H^k(\Upsilon_T)} \leq$

$C_1\nu$, and as a result: $\|\underline{\hat{w}}_\nu - \underline{\hat{w}}\|_{H^k(\Omega_T^-)} \leq C_2\nu$. Using the properties of $\widetilde{g}_\nu^-$, there is some function $\underline{u}^-$ such that:

$$\|\underline{u}_\nu^- - \underline{u}^-\|_{H^k(\Omega_T^-)} \leq C_3\nu,$$

moreover it satisfies:

$$\underline{u}^-|_{x=0} = u|_{x=0}.$$

Considering now the difference $\underline{u}_\nu - u$, it is solution of the well-posed mixed hyperbolic problem:

$$\begin{cases} \mathcal{H}(\underline{u}_\nu - u) = f_\nu - f, \quad \{x > 0\}, \\ \Gamma(\underline{u}_\nu - u)|_{x=0} = \Gamma(g_\nu - g), \\ (\underline{u}_\nu - u)|_{t<0} = 0 \quad . \end{cases}$$

Since this problem satisfies a uniform Lopatinski condition, and exploiting the definition of f_ν and g_ν, there is $c > 0$ such that:

$$\|\underline{u}_\nu - u\|_{H^k(\Omega_T^+)} \leq c\nu.$$

Moreover we have:

$$\|\underline{u}_\nu^\varepsilon - u\|^2_{H^k(\Omega_T^+)} \leq \|\underline{u}_\nu^\varepsilon - \underline{u}_\nu\|^2_{H^k(\Omega_T^+)} + \|\underline{u}_\nu - u\|^2_{H^k(\Omega_T^+)},$$

and

$$\|\underline{u}_\nu^\varepsilon - \underline{u}^-\|^2_{H^k(\Omega_T^-)} \leq \|\underline{u}_\nu^\varepsilon - \underline{u}_\nu^-\|^2_{H^k(\Omega_T^-)} + \|\underline{u}_\nu^- - \underline{u}^-\|^2_{H^k(\Omega_T^-)},$$

hence there are two positive constants c and C_ν such that:

$$\|\underline{u}_\nu^\varepsilon - u\|^2_{H^k(\Omega_T^+)} + \|\underline{u}_\nu^\varepsilon - \underline{u}^-\|^2_{H^k(\Omega_T^-)} \leq c\nu^2 + C_\nu\varepsilon^2.$$

Let us fix $\delta > 0$, we obtain then, for small enough $\varepsilon > 0$:

$$\|\underline{u}_\nu^\varepsilon - u\|^2_{H^k(\Omega_T^+)} + \|\underline{u}_\nu^\varepsilon - \underline{u}^-\|^2_{H^k(\Omega_T^-)} \leq \delta,$$

by taking for instance $\nu^2 = \frac{\delta}{2c}$. Considering now ν as a continuous function of ε yields:

$$\|\underline{u}_{\nu(\varepsilon)}^\varepsilon - u\|^2_{H^k(\Omega_T^+)} + \|\underline{u}_{\nu(\varepsilon)}^\varepsilon - \underline{u}^-\|^2_{H^k(\Omega_T^-)} \leq c\left(\nu(\varepsilon)\right)^2 + C_{\nu(\varepsilon)}\varepsilon^2.$$

So, considering the functions ν such that:

$$\lim_{\varepsilon \to 0^+} \nu(\varepsilon) = 0,$$

and

$$\lim_{\varepsilon \to 0^+} C_{\nu(\varepsilon)} \varepsilon^2 = 0,$$

we obtain Theorem 6.1.11 and Theorem 6.1.12. Note that the Assumption

$$\lim_{\varepsilon \to 0^+} C_{\nu(\varepsilon)} \varepsilon^2 = 0$$

ensures the rate of convergence of ν towards 0, is not too fast.

6.4 Proof of Theorem 6.1.13, Theorem 6.1.16, and Theorem 6.1.17.

Like in the proof of Theorem 6.1.6, we begin by constructing formally an approximate solution of equation (6.1.7). We prove then suitable energy estimates that ensures both u^ε and its approximate solution converges towards $\widetilde{u}$ as $\varepsilon \to 0^+$. We establish then Theorem 6.1.16 and Theorem 6.1.17 relying on the proved stability estimates.

6.4.1 Construction of an approximate solution.

The goal of this Lemma is to replace the boundary condition $\Gamma u|_{x=0} = \Gamma g$ of problem (6.1.1) by a condition of the form $\underline{\mathbf{P}}^-(e^{-\gamma t} u)|_{x=0} = h$ with a suitable $h \in H^k(\Upsilon_T)$.

Lemma 6.4.1. *Let u denote the unique solution in $H^k(\Omega_T^+)$ of the mixed hyperbolic problem (6.1.1), $\underline{\mathbf{P}}^+(\partial_t, \partial_y, \gamma)\, (e^{-\gamma t} u)$ does not depend of the choice of the boundary operator Γ and of g. Let us introduce the function h of $H^k(\Upsilon_T)$ defined by:*

$$\underline{\mathbf{P}}^- \left(e^{-\gamma t} v|_{x=0}\right) + \underline{\mathbf{\Pi}} \left(e^{-\gamma t} (g - v|_{x=0})\right).$$

The solution u of the mixed hyperbolic problem (6.1.1) is the unique solution of the following well-posed mixed hyperbolic problem (6.4.1):

$$\text{(6.4.1)} \qquad \begin{cases} \mathcal{H} u = f, \quad \{x > 0\}, \\ \underline{\mathbf{P}}^-(\partial_t, \partial_y, \gamma) \left(e^{-\gamma t} u|_{x=0}\right) = h, \\ u|_{t<0} = 0 \quad . \end{cases}$$

In addition, the mapping $(f, g) \to h$ is linear continuous from $H^k(\Omega_T^+) \times H^k(\Upsilon_T)$ to $H^k(\Upsilon_T)$.

Proof. Let v denote a solution in $H^k(\Omega_T)$ of the equation:

$$\begin{cases} \mathcal{H}v = f, & (t, y, x) \in \Omega_T, \\ v|_{t<0} = 0 & . \end{cases}$$

We introduce then $\mathbb{U}$ which is, by definition, the solution of the following mixed hyperbolic problem:

$$\begin{cases} \mathcal{H}\mathbb{U} = 0, \quad \{x > 0\}, \\ \underline{\Gamma}(\partial_t, \partial_y, \gamma)\mathbb{U}|_{x=0} = \underline{\Gamma}(\partial_t, \partial_y, \gamma)g - \underline{\Gamma}(\partial_t, \partial_y, \gamma)v|_{x=0}, \\ \mathbb{U}|_{t<0} = 0 \quad . \end{cases}$$

The right hand side of the boundary condition is, a priori, in $H^{k-\frac{1}{2}}(\Upsilon_T)$. Hence the solution $\mathbb{U}$ belongs to $H^{k-\frac{1}{2}}(\Omega_T^+)$. By construction we have:

$$u = \mathbb{U} + v. \tag{6.4.2}$$

Hence, since $u \in H^k(\Omega_T^+)$ and $v \in H^k(\Omega_T^+)$, in fact we have:

$$\mathbb{U} \in H^k(\Omega_T^+).$$

Let $\hat{\mathbb{U}}$ denote the Fourier-Laplace transform in (t, y) of $\mathbb{U}$ (Fourier-Laplace transform tangential to the boundary) given by: $\mathcal{F}(e^{-\gamma t}\mathbb{U})$. It satisfies the following symbolic equation:

$$\begin{cases} \partial_x \hat{\mathbb{U}} = A(\zeta)\hat{\mathbb{U}}, \quad \{x > 0\}, \\ \Gamma(\zeta)\hat{\mathbb{U}}|_{x=0} = \Gamma(\zeta)\hat{g} - \Gamma(\zeta)\hat{v}|_{x=0}, \end{cases}$$

where $\hat{g}$ and $\hat{v}$ denotes respectively the tangential Fourier-Laplace transform of g and v. Since $A(\zeta)$ is independent of x, projecting the above equation on $\mathbb{E}_+(A(\zeta))$ gives then:

$$\partial_x \mathbf{P}^+ \hat{\mathbb{U}} = A(\zeta)\mathbf{P}^+ \hat{\mathbb{U}}.$$

Moreover $\mathbf{P}^+\hat{\mathbb{U}}|_{x=0} \in \mathbb{E}_-(A(\zeta)) \bigcap \mathbb{E}_+(A(\zeta))$ since $\lim_{x\to\infty} \mathbf{P}^+\hat{\mathbb{U}} = 0$. Hence, there holds:

$$\mathbf{P}^+\hat{\mathbb{U}} = 0,$$

and thus

$$\hat{\mathbb{U}} = \mathbf{P}^{-}\hat{\mathbb{U}}.$$

The boundary condition:

$$\Gamma(\zeta)\hat{\mathbb{U}}|_{x=0} = \Gamma(\zeta)\hat{g} - \Gamma(\zeta)\hat{v}|_{x=0}$$

is equivalent to:

$$\hat{\mathbb{U}}|_{x=0} \in \hat{g} - \hat{v}|_{x=0} + \mathrm{Ker}\Gamma.$$

We have thus:

$$\mathbf{P}^{-}\hat{\mathbb{U}}|_{x=0} \in \hat{g} - \hat{v}|_{x=0} + \ker\Gamma.$$

Let us denote by $\mathbf{\Pi}$ the projector on $\widetilde{\mathbb{E}}_{-}(A)$ parallel to $\ker\Gamma$, which has a sense because the Uniform Lopatinski Condition holds.
Since $\hat{\mathbb{U}}|_{x=0} \in \widetilde{\mathbb{E}}_{-}(A)$, and of the Uniform Lopatinski Condition, the above boundary condition is equivalent to:

$$\mathbf{\Pi}\mathbf{P}^{-}\hat{\mathbb{U}}|_{x=0} = \mathbf{\Pi}(\hat{g} - \hat{v}|_{x=0}),$$

and thus, because $\mathbf{P}^{-}\hat{\mathbb{U}}|_{x=0}$ belongs to $\mathbb{E}_{-}(A)$, we have:

$$\mathbf{P}^{-}\hat{\mathbb{U}}|_{x=0} = \mathbf{\Pi}(\hat{g} - \hat{v}|_{x=0}).$$

As a consequence, we obtain:

$$\mathbf{P}^{-}\hat{u}|_{x=0} = \mathbf{P}^{-}\hat{v}|_{x=0} + \mathbf{\Pi}(\hat{g} - \hat{v}|_{x=0}).$$

Hence, there holds:

$$\underline{\mathbf{P}}^{-}\left(e^{-\gamma t}u|_{x=0}\right) = \underline{\mathbf{P}}^{-}\left(e^{-\gamma t}v|_{x=0}\right) + \underline{\mathbf{\Pi}}\left(e^{-\gamma t}(g - v|_{x=0})\right) := h.$$

$\underline{\mathbf{P}}^{+}(\partial_t, \partial_y, \gamma)\left(e^{-\gamma t}u\right) = \underline{\mathbf{P}}^{+}(\partial_t, \partial_y, \gamma)\left(e^{-\gamma t}v\right)$, thus it does not depend of the choice of the boundary operator Γ and of g. Moreover, since $u|_{x=0} \in H^k(\Upsilon_T)$, it follows that
$g \in H^k(\Upsilon_T)$. Now, since the Uniform Lopatinski Condition holds, u satisfies the following energy estimate:

$$\frac{1}{\gamma}\|u\|^2_{e^{\gamma t}L^2(\Omega_T^+)} + \|u|_{x=0}\|^2_{e^{\gamma t}L^2(\Upsilon_T)} \leq \gamma\|f\|^2_{e^{\gamma t}L^2(\Omega_T^+)} + \|g\|_{e^{\gamma t}L^2(\Upsilon_T)},$$

More generally, we have:

$$\frac{1}{\gamma}\|u\|^2_{H^k_\gamma(\Omega_T^+)} + \|u|_{x=0}\|^2_{H^k_\gamma(\Upsilon_T)} \leq \gamma\|f\|^2_{H^k_\gamma(\Omega_T^+)} + \|g\|_{H^k_\gamma(\Upsilon_T)}.$$

where $\|\varpi\|^2_{H^k_\gamma} := \sum_{|\alpha|=0}^k \gamma^{k-|\alpha|}\|\partial^\alpha \varpi\|^2_{e^{\gamma t}L^2}$.
$h = \underline{\mathbf{P}}^-(e^{-\gamma t}u|_{x=0})$ hence

$$\|h\|^2_{L^2(\Upsilon_T)} \leq C\|e^{-\gamma t}u|_{x=0}\|^2_{L^2(\Upsilon_T)} = C\|u|_{x=0}\|^2_{e^{\gamma t}L^2(\Upsilon_T)};$$

and for $0 \leq j \leq d-1$, there holds:

$$\|\partial_j h\|^2_{L^2(\Upsilon_T)} \leq c_j\|\eta_j \mathbf{P}^- \mathcal{F}(e^{-\gamma t}u)|_{x=0}\| \leq c_j'\|u|_{x=0}\|_{H^1_\gamma(\Upsilon_T)}.$$

More generally, we have:

$$\|h\|^2_{H^k_\gamma(\Upsilon_T)} \leq C_k\gamma\|f\|^2_{H^k_\gamma(\Omega_T^+)} + C_k\|g\|^2_{H^k_\gamma(\Upsilon_T)}.$$

But γ is a positive real number fixed once and for all at the beginning of the paper, hence this proves that the mapping $(f, g) \to h$ is continuous from
$H^k(\Omega_T^+) \times H^k(\Upsilon_T)$ to $H^k(\Upsilon_T)$. $\square$ As we will see, Lemma 6.4.1 is central in our construction of an approximate solution. We will construct an approximate solution

$$u^\varepsilon_{app} := u^{\varepsilon+}_{app}\mathbf{1}_{x>0} + u^{\varepsilon-}_{app}\mathbf{1}_{x<0},$$

along the following ansatz:

$$u^{\varepsilon+}_{app} := \sum_{j=0}^M \varepsilon^j u_j^+(t, y, x),$$

with $u_j^+ \in H_\gamma^{k-\frac{3}{2}j}(\Omega_T^+)$, $u_j^+|_{x=0} \in H_\gamma^{k-\frac{3}{2}j}(\Upsilon_T)$; and

$$u^{\varepsilon-}_{app} := \sum_{j=0}^M \varepsilon^j u_j^-(t, y, x),$$

with $u_j^- \in H_\gamma^{k-\frac{3}{2}j}(\Omega_T^-)$, $u_j^-|_{x=0} \in H_\gamma^{k-\frac{3}{2}j}(\Upsilon_T)$. As usual, we will refer to the terms $u_j^\pm$ as profiles. We will rather work on the reformulation of problem (6.1.7) as the transmission problem (6.4.3):

$$(6.4.3) \qquad \begin{cases} \mathcal{H}u^{\varepsilon+} = f, \quad \{x > 0\}, \\ \mathcal{H}u^{\varepsilon-} + \dfrac{1}{\varepsilon}A_d e^{\gamma t}\underline{\mathbf{P}}^- e^{-\gamma t}u^{\varepsilon-} = \dfrac{1}{\varepsilon}A_d e^{\gamma t}\widetilde{h}, \quad \{x < 0\}, \\ u^{\varepsilon+}|_{x=0} - u^{\varepsilon-}|_{x=0} = 0, \\ u^{\varepsilon\pm}|_{t<0} = 0 \quad . \end{cases}$$

Plugging $u_{app}^{\varepsilon+}$ and $u_{app}^{\varepsilon-}$ in (6.4.3) and identifying the terms with same power in ε, we obtain the following profiles equations:

- Identification of the terms of order ε^{-1} :

$$A_d e^{\gamma t} \underline{\mathbf{P}}^{-} e^{-\gamma t} u_0^{-} = A_d e^{\gamma t} \widetilde{h}, \quad \{x < 0\}. \tag{6.4.4}$$

- Identification of the terms of order ε^{0} :

$$\mathcal{H} u_0^{-} + A_d e^{\gamma t} \underline{\mathbf{P}}^{-} e^{-\gamma t} u_1^{-} = 0, \quad \{x < 0\}. \tag{6.4.5}$$

$$\mathcal{H} u_0^{+} = f, \quad \{x > 0\}, \tag{6.4.6}$$

- Identification of the terms of order ε^{j} for $j \geq 1$:

$$\mathcal{H} u_j^{-} + A_d e^{\gamma t} \underline{\mathbf{P}}^{-} e^{-\gamma t} u_{j+1}^{-} = 0, \quad \{x < 0\}. \tag{6.4.7}$$

$$\mathcal{H} u_j^{+} = 0, \quad \{x > 0\}, \tag{6.4.8}$$

- Translation of the continuity condition over the boundary on the profiles:

  For all $1 \leq j \leq M$, there holds:

$$u_j^{+}|_{x=0} - u_j^{-}|_{x=0} = 0. \tag{6.4.9}$$

Denote by $\hat{u}_j^{\pm} := \mathcal{F}(e^{-\gamma t} u_j^{\pm})$. We have then:

$$u_j^{\pm} := e^{\gamma t} \mathcal{F}^{-1}(\hat{u}_j^{\pm}).$$

We will now give the equations satisfied by the Fourier-Laplace transform of the profiles: $\hat{u}_j^{\pm}$. First, equation (6.4.4) is equivalent to:

$$\mathbf{P}^{-} \hat{u}_0^{-} = \mathcal{F}(\widetilde{h}), \quad \{x < 0\}.$$

We deduce from this equation that there holds:

$$\mathbf{P}^- \hat{u}_0^-|_{x=0} = \hat{h}.$$

Then, using (6.4.9) for $j = 0$, and (6.4.6) gives that, for γ big enough,

$$u_0^+ = \mathcal{F}(e^{-\gamma t}\hat{u}_0^+),$$

where $\hat{u}_0^+$ is the solution of the well-posed first order ODE in x:

$$\begin{cases} \partial_x \hat{u}_0^+ - A\hat{u}_0^+ = \mathcal{F}(e^{-\gamma t}(A_d)^{-1} f), \quad \{x > 0\}, \\ \mathbf{P}^- \hat{u}_0^+|_{x=0} = h, \end{cases}$$

Thus u_0^+ is solution of:

$$\begin{cases} \mathcal{H} u_0^+ = f, \quad \{x > 0\}, \\ e^{\gamma t} \underline{\mathbf{P}}^- e^{-\gamma t} u_0^+|_{x=0} = h. \end{cases}$$

Thanks to Lemma 6.4.1, we recognize u_0^+ as the solution of our starting mixed hyperbolic problem (6.1.1). Once u_0^+ is known, so is $\hat{u}_0^+$ and thus $\hat{u}_0^-|_{x=0}$ is given by:

$$\hat{u}_0^-|_{x=0} = \hat{u}_0^+|_{x=0}.$$

Moreover,

$$u_0^+|_{x=0} = u_0^-|_{x=0} \in H^k_\gamma(\Upsilon_T).$$

By (6.4.5), there holds:

$$\partial_x \hat{u}_0^- - A\hat{u}_0^- + \mathbf{P}^- \hat{u}_1^- = 0, \quad \{x < 0\}.$$

As a consequence, $\mathbf{P}^+ \hat{u}_0^-$ is given by the well-posed ODE:

$$\begin{cases} \partial_x (\mathbf{P}^+ \hat{u}_0^-) - A(\mathbf{P}^+ \hat{u}_0^-) = 0, \quad \{x < 0\}, \\ \mathbf{P}^+ \hat{u}_0^-|_{x=0} = \mathbf{P}^+ \hat{u}_0^+|_{x=0}. \end{cases}$$

Indeed, since $\ker \mathbf{P}^+(\zeta) = \mathbb{E}_-(A(\zeta))$, this problem satisfies the Uniform Lopatinski Condition: for all $\zeta \neq 0$, there holds:

$$\mathbb{E}_-(A(\zeta)) \bigoplus \mathbb{E}_+(A(\zeta)) = \mathbb{C}^N.$$

For γ big enough, by linearity of the inverse Fourier transform, u_0^- can then be computed by:

$$u_0^- := e^{\gamma t} \mathcal{F}^{-1}(\mathbf{P}^- \hat{u}_0^-) + e^{\gamma t} \mathcal{F}^{-1}(\mathbf{P}^+ \hat{u}_0^-).$$

Following up with that process of construction, we can go on with the construction of the profiles at any order. Indeed, assume that all the profiles (u_j^+, u_j^-) up to order j have been computed. Then consider the equation obtained through identification:

$$\mathbf{P}^- \hat{u}_{j+1}^- = -\partial_x \hat{u}_j^- + A\hat{u}_j^-, \quad \{x < 0\}.$$

We see there is a loss of regularity between $\hat{u}_{j+1}^-$ and $\hat{u}_j^-$.
Let us say that $u_j^\pm \in H_\gamma^{m_j}(\Omega_T^\pm)$. Considering the traces, we have then: $u_j^\pm|_{x=0} \in H_\gamma^{m_j-\frac{1}{2}}(\Upsilon_T)$. We will show in this part how the Sobolev regularity of the profiles $u_{j+1}^\pm$, which is by definition m_{j+1}, can be computed knowing m_j. To begin with $\underline{\mathbf{P}}^- u_{j+1}^-$ belongs to $H_\gamma^{m_j-1}(\Omega_T^-)$. $\underline{\mathbf{P}}^- u_{j+1}^+|_{x=0}$, which belongs to $H_\gamma^{m_j-\frac{3}{2}}(\Upsilon_T)$, is known by $\underline{\mathbf{P}}^- u_{j+1}^+|_{x=0} = e^{\gamma t}\mathcal{F}^{-1}(\mathbf{P}^- \hat{u}_{j+1}^+|_{x=0})$, with:

$$\mathbf{P}^- \hat{u}_{j+1}^+|_{x=0} = \mathbf{P}^- \hat{u}_{j+1}^-|_{x=0}.$$

Hence, $\hat{u}_{j+1}^+ := \mathcal{F}(e^{-\gamma t} u_{j+1}^+)$ is the solution of the first order ODE in x :

$$\begin{cases} \partial_x \hat{u}_{j+1}^+ - A\hat{u}_{j+1}^+ = 0, \quad \{x > 0\}, \\ \mathbf{P}^- \hat{u}_{j+1}^+|_{x=0} = \mathbf{P}^- \hat{u}_{j+1}^-|_{x=0}. \end{cases}$$

Since $\ker \mathbf{P}^-(\zeta) = \mathbb{E}_+(A(\zeta))$, this problem satisfies the Uniform Lopatinski Condition: for all $\zeta \neq 0$, there holds:

$$\mathbb{E}_-(A(\zeta)) \bigoplus \mathbb{E}_+(A(\zeta)) = \mathbb{C}^N.$$

As a consequence, this problem is well-posed and, $u_{j+1}^+ \in H_\gamma^{m_j-\frac{3}{2}}(\Omega_T^+)$. Moreover, there holds:

$$u_{j+1}^+|_{x=0} = u_{j+1}^-|_{x=0} \in H_\gamma^{m_j-\frac{3}{2}}(\Upsilon_T).$$

Indeed, $\mathbf{P}^+ \hat{u}_{j+1}^+ \in H^\infty(\mathbb{R}_+^{d+1})$ hence $\underline{\mathbf{P}}^+ u_{j+1}^+|_{x=0} \in H_\gamma^{m_j-\frac{3}{2}}(\Upsilon_T)$ and thus $u_{j+1}^+|_{x=0} \in H_\gamma^{m_j-\frac{3}{2}}(\Upsilon_T)$. Furthermore, we have:

$$u_{j+1}^-|_{x=0} = u_{j+1}^+|_{x=0}.$$

Applying $\mathbf{P}^+$ on the following equation:

$$\mathbf{P}^- \hat{u}^-_{j+2} = -\partial_x \hat{u}^-_{j+1} + A\hat{u}^-_{j+1}, \quad \{x<0\};$$

we obtain then the equation:

$$\partial_x(\mathbf{P}^+\hat{u}^-_{j+1}) - A\mathbf{P}^+\hat{u}^-_{j+1} = 0, \quad \{x<0\}.$$

Remark 6.4.2.

$$\mathbf{P}^- \hat{u}^-_{j+2} = -\partial_x \hat{u}^-_{j+1} + A\hat{u}^-_{j+1}, \quad \{x<0\}.$$

shows that the 'Fourier profile' $\hat{u}^-_{j+1}$ must be so that $-\partial_x \hat{u}^-_{j+1} + A\hat{u}^-_{j+1}$ is polarized on $\mathbb{E}_-(A)$. It is indeed the case because we search for $\hat{u}^-_{j+1}$ satisfying:

$$\partial_x(\mathbf{P}^+\hat{u}^-_{j+1}) - A\mathbf{P}^+\hat{u}^-_{j+1} = 0, \quad \{x<0\}.$$

u^-_{j+1} is given by:

$$u^-_{j+1} := e^{\gamma t}\mathcal{F}^{-1}(\mathbf{P}^-\hat{u}^-_{j+1}) + e^{\gamma t}\mathcal{F}^{-1}(\mathbf{P}^+\hat{u}^-_{j+1}).$$

with $\underline{\mathbf{P}}^+ u^-_{j+1} = e^{\gamma t}\mathcal{F}^{-1}(\mathbf{P}^+\hat{u}^-_{j+1})$ belongs to $H^{m_j-\frac{3}{2}}_\gamma(\Omega^+_T)$ and is the unique solution of the well-posed first order ODE:

$$\begin{cases} \partial_x(\mathbf{P}^+\hat{u}^-_{j+1}) - A(\mathbf{P}^+\hat{u}^-_{j+1}) = 0, \quad \{x<0\}, \\ \mathbf{P}^+\hat{u}^-_{j+1}|_{x=0} = \mathbf{P}^+\hat{u}^+_{j+1}|_{x=0}. \end{cases}$$

The profile u^-_{j+1} belongs to $H^{m_j-\frac{3}{2}}_\gamma(\Omega^-_T)$. This achieves to show that the knowledge of (u^+_j, u^-_j), allows us to compute (u^+_{j+1}, u^-_{j+1}).
Moreover $m_{j+1} = m_j - \frac{3}{2}$, that is to say that a construction of each supplementary profile consummate $\frac{3}{2}$ of Sobolev regularity. In practice, we take:

$$u^{\varepsilon+}_{app} = u^+_0,$$

$$u^{\varepsilon-}_{app} = u^-_0 + \varepsilon u^-_1.$$

As a result, the approximate solution writes $u^\varepsilon_{app} := u^{\varepsilon+}_{app}\mathbf{1}_{x>0} + u^{\varepsilon-}_{app}\mathbf{1}_{x<0}$; where $u^{\varepsilon+}_{app}$ belongs to $H^k_\gamma(\Omega^+_T)$ and $u^{\varepsilon-}_{app}$ belongs to $H^{k-\frac{3}{2}}_\gamma(\Omega^-_T)$. The so defined u^ε_{app} is solution of a well-posed problem of the form:
(6.4.10)

$$\begin{cases} \mathcal{H}u^\varepsilon_{app} + \dfrac{1}{\varepsilon}A_d e^{\gamma t}\underline{\mathbf{P}}^- e^{-\gamma t} u^\varepsilon_{app}\mathbf{1}_{x<0} = f\mathbf{1}_{x>0} + \dfrac{1}{\varepsilon}A_d e^{\gamma t}\widetilde{h}\mathbf{1}_{x<0} + \varepsilon r^\varepsilon, \\ u^\varepsilon_{app}|_{t<0} = 0 \quad . \end{cases}$$

Where $r^\varepsilon := r^{\varepsilon+}\mathbf{1}_{x>0} + r^{\varepsilon-}\mathbf{1}_{x<0}$, with $r^{\varepsilon+} \in H_\gamma^{k-\frac{5}{2}}(\Omega_T^+)$ and $r^{\varepsilon-} \in H_\gamma^{k-3}(\Omega_T^-)$.

6.4.2 Asymptotic Stability of the problem as ε tends towards zero.

Denote by $v^\varepsilon = u^\varepsilon_{app} - u^\varepsilon$. By construction of u^ε_{app}, v^ε is solution of the following Cauchy problem:

$$\begin{cases} \mathcal{H}v^\varepsilon + \dfrac{1}{\varepsilon} A_d e^{\gamma t}\underline{\mathbf{P}}^- e^{-\gamma t} v^\varepsilon \mathbf{1}_{x<0} = \varepsilon r^\varepsilon, \\ v^\varepsilon|_{t<0} = 0 \quad . \end{cases} \tag{6.4.11}$$

For all positive ε, this problem is well-posed. In order to investigate the stability of this problem as ε goes to zero, we will reformulate it as a transmission problem. The restrictions of v^ε to $\{x > 0\}$ and $\{x < 0\}$, respectively denoted by $v^{\varepsilon+}$ and $v^{\varepsilon-}$ are solution the following transmission problem:

$$\begin{cases} \mathcal{H}v^{\varepsilon+} = \varepsilon r^{\varepsilon+}, \quad \{x > 0\}, \\ \mathcal{H}v^{\varepsilon-} + \dfrac{1}{\varepsilon} A_d e^{\gamma t}\underline{\mathbf{P}}^- e^{-\gamma t} v^{\varepsilon-} = \varepsilon r^{\varepsilon-}, \quad \{x < 0\}, \\ v^{\varepsilon+}|_{x=0} - v^{\varepsilon-}|_{x=0} = 0, \\ v^{\varepsilon\pm}|_{t<0} = 0 \quad . \end{cases} \tag{6.4.12}$$

Let us denote by V^ε the function, valued in $\mathbb{R}^{2N}$, defined for all $\{x > 0\}$ and $(t, y) \in [0, T] \times \mathbb{R}^{d-1}$ by:

$$V^\varepsilon(t, y, x) = \begin{pmatrix} V^{\varepsilon+}(t, y, x) \\ V^{\varepsilon-}(t, y, -x) \end{pmatrix}.$$

v^ε is solution of the Cauchy problem (6.4.11) iff V^ε is solution of the mixed hyperbolic problem on a half space (6.4.13) given below:

$$\begin{cases} \widetilde{\mathcal{H}}V^\varepsilon + B^\varepsilon V^\varepsilon = \varepsilon R^\varepsilon, \quad \{x > 0\}, \\ \widetilde{\Gamma} V^\varepsilon|_{x=0} = 0, \\ V^\varepsilon|_{t<0} = 0 \quad , \end{cases} \tag{6.4.13}$$

where

$$\widetilde{\mathcal{H}} = \partial_t + \sum_{j=1}^{d-1} \begin{pmatrix} A_j & 0 \\ 0 & A_j \end{pmatrix} \partial_j + \begin{pmatrix} A_d & 0 \\ 0 & -A_d \end{pmatrix} \partial_x,$$

$$B^\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\varepsilon} A_d e^{\gamma t} \underline{\mathbf{P}}^- e^{-\gamma t} \end{pmatrix},$$

$$R^\varepsilon(t,y,x) = \begin{pmatrix} r^{\varepsilon +}(t,y,x) \\ r^{\varepsilon -}(t,y,-x) \end{pmatrix},$$

and

$$\widetilde{\Gamma} = \begin{pmatrix} Id & -Id \end{pmatrix}.$$

Returning to the construction of our approximate solution, we have $R^\varepsilon \in H_\gamma^{k-\frac{5}{2}}(\Omega_T^+) \times H_\gamma^{k-3}(\Omega_T^+)$ and is such that $R^\varepsilon|_{t<0} = 0$.
In fact $R^\varepsilon \in H_\gamma^{k'}(\Omega_T^+)$ with $k' = k-3$. For all positive ε, there exists a unique solution V^ε in $H_\gamma^k(\Omega_T^+)$ to the above problem. We will prove here that this solution converges, uniformly in ε, towards 0 in $H_\gamma^{k'}(\Omega_T^+)$, as ε vanishes. As in the proof of Kreiss Theorem, see [CP81] for instance, existence of solution for mixed hyperbolic systems like (6.1.7) or (6.4.13), are obtained through the proof of both direct and 'dual' a priori estimates on an adjoint problem. This estimates results in the constant coefficient case of estimates on the Fourier-Laplace transform of the solution. Additionally, if this 'Fourier' estimate can be proved, both direct and 'dual' energy estimates are deduced from it. In a first step, let us recall formally how to conduct the Fourier-Laplace transform of a mixed hyperbolic problem:

$$\begin{cases} \mathcal{H}u = f, \quad \{x>0\}, \\ \Gamma u|_{x=0} = g, \\ u|_{t<0} = 0 \quad , \end{cases}$$

Denote by $u_* := e^{-\gamma t} u$, u_* is in particular a solution of the following problem:

$$\begin{cases} \mathcal{H}u_* + \gamma u_* = e^{-\gamma t} f, \quad \{x>0\}, \\ \Gamma u_*|_{x=0} = g \quad . \end{cases}$$

We take then the tangential (with respect to (t,y)) Fourier transform of this equation, which gives:

$$\begin{cases} A_d \partial_x \hat{u}_* + (\gamma + i\tau)\hat{u}_* + i\eta_j \sum_{j=1}^{d-1} A_j \hat{u}_* = \mathcal{F}\left(e^{-\gamma t} f\right), \quad \{x>0\}, \\ \Gamma \hat{u}_*|_{x=0} = \hat{g} \quad . \end{cases}$$

Multiplying this equation by A_d^{-1}, we obtain that u^* is solution of the following ODE in x:

$$\begin{cases} \partial_x \hat{u}_* - A\hat{u}_* = (A_d)^{-1}\mathcal{F}\left(e^{-\gamma t} f\right), & \{x > 0\}, \\ \Gamma \hat{u}_*|_{x=0} = \hat{g} & . \end{cases}$$

Note that, $\hat{u}_*$ and u can be freely deduced from each other through the formulas:

$$\hat{u}_* = \mathcal{F}(e^{-\gamma t} u)$$

and

$$u = e^{\gamma t}\mathcal{F}^{-1}(\hat{u}_*).$$

We shall now introduce a rescaled solution $\underline{V}^\varepsilon$ of the solution V^ε of (6.4.13) defined as follows: $\underline{V}^\varepsilon(t, y, x) := V^\varepsilon(t, y, \varepsilon x)$, and the rescaled remainder: $\underline{R}^\varepsilon(t, y, x) := R^\varepsilon(t, y, \varepsilon x)$. Denoting by $\hat{\underline{V}}^\varepsilon = \mathcal{F}(e^{-\gamma t}\underline{V})$, the associated equation writes then:

$$\begin{cases} \partial_x \hat{\underline{V}}^\varepsilon - \varepsilon \widetilde{A}\hat{\underline{V}}^\varepsilon + M\hat{\underline{V}}^\varepsilon = \varepsilon^2 \hat{R}^\varepsilon, & \{x > 0\}, \\ \widetilde{\Gamma}\hat{\underline{V}}^\varepsilon|_{x=0} = 0 & . \end{cases}$$

where

$$M(\zeta) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{P}^-(\zeta) \end{pmatrix}.$$

We remark that

$$\varepsilon \widetilde{A}(\zeta) = \widetilde{A}(\varepsilon\zeta) = \widetilde{A}(\hat{\zeta}),$$

with $\hat{\zeta} = (\hat{\tau}, \hat{\gamma}, \hat{\eta}) := \varepsilon\zeta$. Moreover $\mathbf{P}^-$ is homogeneous of order zero in ζ. Let us denote $\widetilde{R}^\varepsilon(\hat{\zeta}, x) := \hat{R}^\varepsilon(\zeta, x)$ and $\widetilde{\underline{V}}^\varepsilon(\hat{\zeta}, x) := \hat{\underline{V}}^\varepsilon(\zeta, x)$. Hence $\widetilde{\underline{V}}^\varepsilon$ is solution of the following problem:

$$\begin{cases} \partial_x \widetilde{\underline{V}}^\varepsilon + \left[-\widetilde{A}(\hat{\zeta}) + M(\hat{\zeta})\right] \widetilde{\underline{V}}^\varepsilon = \varepsilon^2 \widetilde{R}^\varepsilon(\hat{\zeta}, x), & \{x > 0\}, \\ \widetilde{\Gamma}\widetilde{\underline{V}}^\varepsilon|_{x=0} = 0 & . \end{cases}$$

As a consequence, the Uniform Lopatinski Condition for problem (6.4.13) writes: For all $\hat{\gamma} > 0$,

$$|det(\mathbb{E}_-(\widetilde{A}(\hat{\zeta}) - M(\hat{\zeta}), \ker\Gamma)| \geq C > 0.$$

In view of the proof of the Proposition (6.4.3), we recall that the spaces $\mathbb{E}_\pm(A)$ have to be considered in the extended sense defined above.

Proposition 6.4.3. *Since $\mathcal{H}$ satisfies the hyperbolicity Assumption in Assumption 6.1.1, the Uniform Lopatinski Condition is satisfied for our present problem; that is to say that, for all $\hat{\zeta}$ such that $\hat{\gamma} > 0$ there holds:*

$$|det(\mathbb{E}_-(\widetilde{A}(\hat{\zeta}) - M(\hat{\zeta}), \ker \Gamma)| \geq C > 0.$$

Proof. We will begin to show that the Uniform Lopatinski Condition writes as well that for all $\hat{\zeta} \neq 0$ there holds:

$$\mathbb{E}_+(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta})) \bigcap \mathbb{E}_-(A(\hat{\zeta})) = \{0\} \quad . \tag{6.4.14}$$

This notation keeps a sense for $\hat{\zeta}$ such that $\hat{\gamma} = 0$ because we will prove a posteriori that the involved linear subspaces continuously extends from $\{\hat{\zeta}, \hat{\gamma} > 0\}$ to $\{\hat{\zeta}, \hat{\gamma} = 0\}$. Then we will prove that, for all $\hat{\zeta}$, the property 6.4.14 holds true. The Uniform Lopatinski Condition writes actually, for all $\hat{\zeta} \neq 0$:

$$\mathbb{E}_-(\widetilde{A}(\hat{\zeta}) - M(\hat{\zeta})) \bigcap \ker \widetilde{\Gamma} = \{0\}.$$

and thus, since we have:

$$\mathbb{E}_-(\widetilde{A}(\hat{\zeta}) - M(\hat{\zeta})) = \mathbb{E}_-(A(\hat{\zeta})) \times \mathbb{E}_+(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta})),$$

and by definition of $\widetilde{\Gamma}$, the Uniform Lopatinski Condition writes then that, for all $\hat{\zeta} \neq 0$, there holds:

$$\mathbb{E}_+(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta})) \bigcap \mathbb{E}_-(A(\hat{\zeta})) = \{0\}.$$

Lemma 6.4.4.

$$\mathbb{E}_- \left(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta})\right) = \mathbb{E}_- \left(A(\hat{\zeta})\right),$$

$$\mathbb{E}_+ \left(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta})\right) = \mathbb{E}_+ \left(A(\hat{\zeta})\right).$$

Proof. For all $\hat{\zeta} \neq 0$, there is an invertible $N \times N$ matrix with complex coefficients $P(\hat{\zeta})$ such that: $P^{-1}AP$ is trigonal and the diagonal coefficients are sorted by increasing order of their real parts. Let us denote by ν the dimension of $\mathbb{E}_-(A)$. The above matrix P traduces the change of basis from the canonical basis of $\mathbb{C}^N$ into $(v_1, \dots, v_\nu, v_{\nu+1}, \dots, v_N)$, where

$$Span\left((v_k)_{1 \leq k \leq \nu}\right) = \mathbb{E}_-(A),$$

and

$$Span\left((v_k)_{\nu+1\leq k\leq N}\right) = \mathbb{E}_+\left(A\right).$$

Moreover, there holds

$$P^{-1}\mathbf{P}^- P = D$$

where D is the diagonal matrix whose ν first diagonal terms are equal to 1 and the $N-\nu$ last diagonal terms are null.

$$P^{-1}(A-\mathbf{P}^-)P = P^{-1}AP - D.$$

$P^{-1}AP - D$ is also trigonal, with the same eigenvalues with positive real part as $P^{-1}AP$ and the same eigenvalues with negative real part as $P^{-1}AP - Id$. As a consequence, for all $\hat{\zeta} \neq 0$, there holds:

$$\mathbb{E}_-\left(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta})\right) = \mathbb{E}_-\left(A(\hat{\zeta})\right),$$

$$\mathbb{E}_+\left(A(\hat{\zeta}) - \mathbf{P}^-(\hat{\zeta})\right) = \mathbb{E}_+\left(A(\hat{\zeta})\right).$$

□

As a consequence of Lemma 6.4.4, the rescaled Uniform Lopatinski Condition for $\varepsilon > 0, \varepsilon \to 0$ happens to be exactly the same as the one written for bigger positive ε. Indeed, it writes, for all $\hat{\zeta} \neq 0$:

$$\mathbb{E}_+(A(\hat{\zeta}))\bigcap \mathbb{E}_-(A(\hat{\zeta})) = \{0\}.$$

□ The Lopatinski condition is satisfied, and, as a result, the following, uniform in ε, energy estimate holds for $\underline{V}^\varepsilon$, for all $\gamma \geq \gamma_{k'} > 0$:

$$\gamma\|\underline{V}^\varepsilon\|^2_{H^{k'}_\gamma(\Omega_T^+)} + \|\underline{V}^\varepsilon|^2_{x=0}\|_{H^{k'}_\gamma(\Upsilon_T)} \leq \frac{C}{\gamma}\|\varepsilon\underline{R}^\varepsilon\|^2_{H^{k'}_\gamma(\Omega_T^+)} \quad ;$$

which is equivalent to:

$$\gamma\|V^\varepsilon\|^2_{H^{k'}_\gamma(\Omega_T^+)} + \|V^\varepsilon|_{x=0}\|^2_{H^{k'}_\gamma(\Upsilon_T)} \leq \frac{C}{\gamma}\|\varepsilon R^\varepsilon\|^2_{H^{k'}_\gamma(\Omega_T^+)} \quad . \tag{6.4.15}$$

This proves the convergence of V^ε towards zero in $H^{k'}_\gamma(\Omega_T^+)$. The weight γ is fixed beforehand thus, in fact, the solution of (6.4.13) tends to zero in $H^{k'}(\Omega_T^+)$ at a rate at least in $\mathcal{O}(\varepsilon)$.

6.5 End of proof of Theorem 6.1.13.

Let us consider V^ε defined by:

$$V^\varepsilon(t,y,x) := \begin{pmatrix} u^{\varepsilon+}_{app}(t,y,x) - u^{\varepsilon+}(t,y,x) \\ u^{\varepsilon-}_{app}(t,y,-x) - u^{\varepsilon-}(t,y,-x) \end{pmatrix}.$$

This notation is perfectly fine because the so-defined function is solution of an equation of the form (6.4.13). Moreover, thanks to the stability estimate (6.4.15), there is γ_k positive such that, for all $\gamma > \gamma_k$, we have:

$$\gamma \| u^\varepsilon_{app} - u^\varepsilon \|^2_{H^{k-3}_\gamma(\Omega^+_T)} + \gamma \| u^\varepsilon_{app} - u^\varepsilon \|^2_{H^{k-3}_\gamma(\Omega^-_T)} + \| u^\varepsilon_{app} - u^\varepsilon \|^2_{H^{k-3}_\gamma(\Upsilon_T)} \leq \frac{C}{\gamma} \| \varepsilon R^{\varepsilon+} \|^2_{H^{k-3}_\gamma(\Omega^+_T)}.$$

Hence, it follows that:

$$\| u^\varepsilon_{app} - u^\varepsilon \|^2_{H^{k-3}(\Omega^+_T)} + \| u^\varepsilon_{app} - u^\varepsilon \|^2_{H^{k-3}(\Omega^-_T)} = \mathcal{O}(\varepsilon^2).$$

Moreover, by construction of u^ε_{app}, we have:

$$\| u^\varepsilon_{app} - u \|^2_{H^{k-3}(\Omega^+_T)} + \| u^\varepsilon_{app} - u^- \|^2_{H^{k-3}(\Omega^-_T)} = \mathcal{O}(\varepsilon^2).$$

As a result, we obtain that there holds:

$$\| u^\varepsilon - u \|^2_{H^{k-3}(\Omega^+_T)} + \| u^\varepsilon - u^- \|^2_{H^{k-3}(\Omega^-_T)} = \mathcal{O}(\varepsilon^2).$$

This concludes the proof of Theorem 6.1.13.

6.5.1 Proof of Theorem 6.1.16 and Theorem 6.1.17.

As we have proves in Lemma 6.4.1, the solution u of the mixed hyperbolic problem (6.1.1) is also solution of the equivalent mixed hyperbolic problem:

$$\begin{cases} \mathcal{H}u = f, \quad \{x > 0\}, \\ \underline{\mathbf{P}}^-(\partial_t, \partial_y, \gamma)\left(e^{-\gamma t} u|_{x=0}\right) = h, \\ u|_{t<0} = 0 \quad . \end{cases}$$

We recall that, by Assumption, there holds:

$$f \in H^k(\Omega^+_T),$$

and

$$g \in H^k(\Upsilon_T),$$

and that $\widetilde{h}$ is defined by:

$$\widetilde{h} := e^{-x^2} h,$$

with $h = \underline{\mathbf{P}}^-(e^{-\gamma t} v|_{x=0}) + \underline{\mathbf{\Pi}} e^{-\gamma t}(g - v|_{x=0})$. For all $0 < \nu < 1$, there is f_ν in $H^\infty(\Omega_T^+)$ and $\widetilde{h}_\nu \in H^\infty(\Omega_T)$ such that:

$$\|f_\nu - f\|_{H^k(\Omega_T^+)} \leq \nu,$$

and

$$\|\widetilde{h}_\nu - \widetilde{h}\|_{H^k(\Omega_T^-)} + \|\widetilde{h}_\nu - \widetilde{h}\|_{H^k(\Omega_T^+)} + \|h_\nu - h\|_{H^k(\Upsilon_T)} \leq \nu^2.$$

We denote then by u_ν the solution of the mixed hyperbolic problem:

$$\begin{cases} \mathcal{H} u_\nu = f_\nu, \quad \{x > 0\}, \\ \underline{\mathbf{P}}^-(\partial_t, \partial_y, \gamma) \left(e^{-\gamma t} u_\nu|_{x=0} \right) = h_\nu, \\ u_\nu|_{t<0} = 0 \quad . \end{cases}$$

There is m_ν such that

$$\underline{\mathbf{P}}^-(\partial_t, \partial_y, \gamma) e^{-\gamma t} m_\nu = h_\nu.$$

Indeed, we can take:

$$m_\nu = e^{\gamma t} \mathcal{F}^{-1} \left[\mathbf{P}^- \mathcal{F} \left(e^{-\gamma t} v_\nu|_{x=0} \right) + \mathbf{\Pi} \mathcal{F} \left(e^{-\gamma t} (g_\nu - v_\nu|_{x=0}) \right) \right]$$

Where v_ν is the solution of the Cauchy problem:

$$\begin{cases} \mathcal{H} v_\nu = f_\nu, \quad (t, y, x) \in \Omega_T, \\ v_\nu|_{t<0} = 0 \quad . \end{cases}$$

and g_ν belongs to $H^\infty(\Upsilon_T)$ and is such that:

$$\lim_{\nu \to 0} \|g_\nu - g\|_{H^k(\Upsilon_T)} = 0.$$

We define then $\widetilde{m}_\nu$, for all $(t, y, x) \in \Omega_T$ by:

$$\widetilde{m}_\nu = m_\nu e^{-x^2}.$$

The restrictions of $\widetilde{m}_\nu$ to $\pm x > 0$ will be denoted by $\widetilde{m}_\nu^\pm$. u_ν is then also the solution of the mixed hyperbolic problem:

$$\begin{cases} \mathcal{H} u_\nu = f_\nu, \quad \{x > 0\}, \\ \Gamma u_\nu|_{x=0} = \Gamma g_\nu, \\ u_\nu|_{t<0} = 0 \quad . \end{cases}$$

Now consider ω_ν defined for $\{x > 0\}$ by:

$$\omega_\nu := u_\nu - \widetilde{m}_\nu^+,$$

ω_ν is the solution of the following mixed hyperbolic problem satisfying a Uniform Lopatinski Condition:

$$\begin{cases} \mathcal{H}\omega_\nu = f_\nu - \mathcal{H}\widetilde{m}_\nu^+, \quad \{x > 0\}, \\ \underline{\mathbf{P}}^-(\partial_t, \partial_y, \gamma)\left(e^{-\gamma t}\omega_\nu|_{x=0}\right) = 0, \\ \omega_\nu|_{t<0} = 0 \quad . \end{cases}$$

and by $u_\nu^\varepsilon := (\omega_\nu^{\varepsilon+} + \widetilde{m}_\nu^+)\mathbf{1}_{x\geq 0} + (\omega_\nu^{\varepsilon-} + \widetilde{m}_\nu^-)\mathbf{1}_{x<0}$, where

$$\omega_\nu^\varepsilon = \omega_\nu^{\varepsilon+}\mathbf{1}_{x\geq 0} + \omega_\nu^{\varepsilon-}\mathbf{1}_{x<0}$$

is defined as the solution of the Cauchy problem:

$$\begin{cases} \mathcal{H}\omega_\nu^\varepsilon + \dfrac{1}{\varepsilon}A_d e^{\gamma t}\underline{\mathbf{P}}^- e^{-\gamma t}\omega_\nu^\varepsilon\mathbf{1}_{x<0} = (f^\nu - \mathcal{H}\widetilde{m}_\nu^+)\mathbf{1}_{x>0} - \mathcal{H}\widetilde{m}_\nu^-\mathbf{1}_{x<0}, \\ \omega_\nu^\varepsilon|_{t<0} = 0 \quad . \end{cases}$$

As a consequence of the stability estimates we have just proved, we have then, for all fixed $\nu > 0$, $\forall s > 0$:

$$\|\omega_\nu^\varepsilon - \omega_\nu^-\|^2_{H^s(\Omega_T^-)} + \|\omega_\nu^\varepsilon - \omega_\nu\|^2_{H^s(\Omega_T^+)} \leq c_\nu\varepsilon^2,$$

with $\omega_\nu^-|_{x=0} = \omega_\nu|_{x=0}$. Let us define: $u_\nu^- := \omega_\nu^- + \widetilde{m}_\nu^-$, we have thus:

$$\|u_\nu^\varepsilon - u_\nu^-\|^2_{H^s(\Omega_T^-)} + \|u_\nu^\varepsilon - u_\nu\|^2_{H^s(\Omega_T^+)} \leq c_\nu\varepsilon^2,$$

with $\underline{u}_\nu^-|_{x=0} = u_\nu|_{x=0}$. ω_ν^- can be computed by: $\omega_\nu^- = e^{\gamma t}\mathcal{F}^{-1}(\hat{\omega}_\nu^-)$, where $\hat{\omega}_\nu^- = \mathbf{P}^+\hat{\omega}_\nu^-$ is the solution of the Kreiss-symmetrizable problem:

$$\begin{cases} \partial_x\hat{\omega}_\nu^- - A\hat{\omega}_\nu^- = -(A_d)^{-1}\mathcal{F}(e^{-\gamma t}\mathcal{H}\widetilde{m}_\nu^-), \quad \{x < 0\}, \\ \mathbf{P}^+\hat{\omega}_\nu^-|_{x=0} = \mathbf{P}^+\hat{\omega}_\nu|_{x=0}, \end{cases}$$

where $\hat{\omega}_\nu$ stands for the Fourier-Laplace transform of ω_ν. Since ω_ν is solution of a mixed hyperbolic problem satisfying a Uniform Lopatinski Condition, there is $C_1 > 0$ such that: $\|\hat{\omega}_\nu - \hat{\omega}\|_{H^k(\Upsilon_T)} \leq C_1\nu$, and as a result:

$$\|\hat{\omega}_\nu - \hat{\omega}\|_{H^k(\Omega_T^-)} \leq C_2\nu.$$

Using the properties of $\widetilde{m}_\nu^-$, there is some function u^- such that:

$$\|u_\nu^- - u^-\|_{H^k(\Omega_T^-)} \leq C_3\nu,$$

moreover it satisfies:

$$u^-|_{x=0} = u|_{x=0}.$$

Considering now the difference $u_\nu - u$, it is solution of the well-posed mixed hyperbolic problem:

$$\begin{cases} \mathcal{H}(u_\nu - u) = f_\nu - f, \quad \{x > 0\}, \\ \Gamma(u_\nu - u)|_{x=0} = \Gamma(g_\nu - g), \\ (u_\nu - u)|_{t<0} = 0 \quad . \end{cases}$$

Since this problem satisfies a uniform Lopatinski condition, and exploiting the definition of f_ν and h_ν, there is $c > 0$ such that:

$$\|u_\nu - u\|_{H^k(\Omega_T^+)} \leq c\nu.$$

Moreover we have:

$$\|u_\nu^\varepsilon - u\|^2_{H^k(\Omega_T^+)} \leq \|u_\nu^\varepsilon - u_\nu\|^2_{H^k(\Omega_T^+)} + \|u_\nu - u\|^2_{H^k(\Omega_T^+)},$$

and

$$\|u_\nu^\varepsilon - u^-\|^2_{H^k(\Omega_T^-)} \leq \|u_\nu^\varepsilon - u_\nu^-\|^2_{H^k(\Omega_T^-)} + \|u_\nu^- - u^-\|^2_{H^k(\Omega_T^-)},$$

hence there are two positive constants c and C_ν such that:

$$\|u_\nu^\varepsilon - u\|^2_{H^k(\Omega_T^+)} + \|u_\nu^\varepsilon - u^-\|^2_{H^k(\Omega_T^-)} \leq c\nu^2 + C_\nu\varepsilon^2.$$

Let us fix $\delta > 0$, we obtain then, for small enough $\varepsilon > 0$:

$$\|u_\nu^\varepsilon - u\|^2_{H^k(\Omega_T^+)} + \|u_\nu^\varepsilon - u^-\|^2_{H^k(\Omega_T^-)} \leq \delta,$$

by taking for instance $\nu^2 = \frac{\delta}{2c}$. Considering now ν as a continuous function of ε yields:

$$\|u_{\nu(\varepsilon)}^\varepsilon - u\|^2_{H^k(\Omega_T^+)} + \|u_{\nu(\varepsilon)}^\varepsilon - u^-\|^2_{H^k(\Omega_T^-)} \leq c\,(\nu(\varepsilon))^2 + C_{\nu(\varepsilon)}\varepsilon^2.$$

So, considering the functions ν such that:

$$\lim_{\varepsilon \to 0^+} \nu(\varepsilon) = 0,$$

and

$$\lim_{\varepsilon \to 0^+} C_{\nu(\varepsilon)}\varepsilon^2 = 0,$$

we obtain Theorem 6.1.16 and Theorem 6.1.17.

6.6 Appendix: Answer to a question asked in [PCLS05].

Penalization methods are frequently used in numerical simulation of fluid dynamics, when a boundary is involved, for example we can refer to [ABF99] by Angot, Bruneau and Fabrie. Roughly speaking, the main idea of this kind of approach is to immerse the original domain into a geometrically bigger an simpler one called fictitious domain. The main interest is that, for the obtained singularly perturbed problem, the discretization is not boundary-fitted to the original domain.

In [FG07], written with Guès, in view of future applications, the authors give two results concerning the penalization of mixed semi-linear hyperbolic problems with dissipative boundary conditions. The quality of the two methods proposed in [FG07] are compared based on the boundary layers they generate. However, it was not clear whether the boundary layers forming were really detrimental in a numerical point of view.

The goal of this note is then, taking as a basis the numerical study of the convergence made in [PCLS05], to show that the numerical rate of convergence, not as good as awaited, observed in [PCLS05] can be explained by the formation of *boundary layers*.

Like in [PCLS05], we will invistigate the quality of the approximation of the solution U of the 1-D wave equation (6.6.1) by a given method of penalization.

$$\begin{cases} \partial_{tt}U - c^2\partial_{xx}U = 0, \quad (x,t) \in]0,\pi[\times\mathbb{R}^+, \\ U|_{x=0} = U|_{x=\pi} = 0, \\ U|_{t=0}(x) = sin(x), \\ \partial_t U|_{t=0} = 0. \end{cases} \tag{6.6.1}$$

As $\varepsilon \to 0^+$, we analyze the approximation of U by U^ε on the half-space $\{x > 0\}$, where $U^\varepsilon = U^{\varepsilon+}\mathbf{1}_{x>0} + U^{\varepsilon-}\mathbf{1}_{x<0}$ is defined as the solution of the following hyperbolic transmission problem :

$$(6.6.2)\qquad \begin{cases} \partial_{tt}U^{\varepsilon+} - c^2\partial_{xx}U^{\varepsilon+} = 0, \quad (x,t) \in]0,\pi[\times\mathbb{R}^+, \\ \partial_{tt}U^{\varepsilon-} - c^2\partial_{xx}U^{\varepsilon-} + \dfrac{1}{\varepsilon^2}U^{\varepsilon-} = 0, \quad (x,t) \in]-\infty,0[\times\mathbb{R}^+, \\ U^{\varepsilon+}|_{x=0} - U^{\varepsilon-}|_{x=0} = 0, \\ \partial_x U^{\varepsilon+}|_{x=0} - \partial_x U^{\varepsilon-}|_{x=0} = 0, \\ U^{\varepsilon+}|_{x=\pi} = 0, \\ U^{\varepsilon\pm}|_{t=0}(x) = sin(x), \quad \{\pm x > 0\}, \\ \partial_t U^{\varepsilon\pm}|_{t=0} = 0, \quad \{\pm x > 0\}. \end{cases}$$

We prove the following result, observed numerically in [PCLS05] :

Theorem 6.6.1. *For all* $0 < \varepsilon < 1$ *and* $T > 0$ *there holds:*

$$\|U^\varepsilon - U\|_{L^\infty(]0,T[:L^2(]-\infty,\pi[)} = \mathcal{O}(\varepsilon^{\frac{1}{2}}).$$

The proof of this theorem incorporates an asymptotic analysis of the boundary layers forming, at any order.

6.6.1 Proof of Theorem 6.6.1

We will now construct formally an approximate solution $(U^{\varepsilon+}_{app}, U^{\varepsilon-}_{app})$ of the solution $(U^{\varepsilon+}, U^{\varepsilon-})$ of the transmission problem (6.6.2). We shall construct this approximate along the following ansatz:

$$U^{\varepsilon+}_{app} = \sum_{j=0}^{M} U^+_j(t,x)\varepsilon^j,$$

$$U^{\varepsilon-}_{app} = \sum_{j=0}^{M} U^-_j\left(t, x, \frac{x}{\varepsilon}\right)\varepsilon^j,$$

where the profiles $U^-_j(t,x,z) := \underline{U}^-_j(t,x) + U^{*-}_j(t,z)$, with

$$\lim_{z\to-\infty} e^{-\alpha z}U^{*-}_j = 0,$$

for some $\alpha > 0$. The layer profiles U^{*-}_j serve the purpose of describing quick fluctuations of the solution as $\varepsilon \to 0^+$.

Since the stability estimates are trivial here, we will only focus on the construction of

$$U^{\varepsilon}_{app} := U^{\varepsilon+}_{app}\mathbf{1}_{x>0} + U^{\varepsilon-}_{app}\mathbf{1}_{x<0}.$$

Plugging $U^{\varepsilon\pm}_{app}$ into problem (6.6.2) and identifying the terms with same power of ε, we obtain the following equations:

$$\underline{U_0^-} = 0,$$

moreover, $U_0^{*-} = 0$ as it is the only solution of the problem:

$$\begin{cases} U_0^{*-} - c^2\partial_{zz}U_0^{*-} = 0, \quad \{z<0\}, \\ \partial_z U_0^{*-}|_{z=0} = 0, \\ \lim\limits_{z\to-\infty} U_0^{*-} = 0. \end{cases}$$

The function $U^{\varepsilon+}_{app}$ converges towards U_0^+ as $\varepsilon \to 0^+$. As awaited, U_0^+ is the solution of the well-posed 1-D wave equation:

$$\begin{cases} \partial_{tt}U_0^+ - c^2\partial_{xx}U_0^+ = 0, \quad (x,t) \in]0,\pi[\times\mathbb{R}^+, \\ U_0^+|_{x=0} = \underline{U_0^-}|_{x=0} + U_0^{*-}|_{z=0} = 0, \\ U_0^+|_{x=\pi} = 0, \\ U_0^+|_{t=0}(x) = sin(x), \quad \{x>0\}, \\ \partial_t U_0^+|_{t=0} = 0, \quad \{x>0\}. \end{cases}$$

Let us now proceed with the construction of the next profiles. First, remark that, not only $\underline{U_0^-} = 0$, but for all $j \geq 1$, there holds:

$$\underline{U_j^-} = 0.$$

The profile U_1^{*-} satisfies the well-posed equation:

$$\begin{cases} U_1^{*-} - c^2\partial_{zz}U_1^{*-} = -\partial_{tt}U_0^{*-} = 0, \quad \{z<0\}, \\ \partial_z U_1^{*-}|_{z=0} = \partial_x U_0^+|_{x=0}, \\ \lim\limits_{z\to-\infty} U_1^{*-} = 0, \end{cases}$$

as a result, we get that:

$$U_1^{*-} = c\partial_x U_0^+|_{x=0} e^{\frac{z}{c}}.$$

We will now prove, by induction, that the the construction of the profiles can go on at any order, which means that for all $M \in \mathbb{N}$ fixed beforehand, we are able to construct U^{ε}_{app} satisfying :

$$
\begin{cases}
\partial_{tt} U^{\varepsilon +}_{app} - c^2 \partial_{xx} U^{\varepsilon +}_{app} = \varepsilon^M R^{\varepsilon +}, \quad (x,t) \in]0,\pi[\times \mathbb{R}^+, \\
\partial_{tt} U^{\varepsilon -}_{app} - c^2 \partial_{xx} U^{\varepsilon -}_{app} + \dfrac{1}{\varepsilon^2} U^{\varepsilon -}_{app} = \varepsilon^M R^{\varepsilon -}, \quad (x,t) \in]-\infty, 0[\times \mathbb{R}^+, \\
U^{\varepsilon +}_{app}|_{x=0} - U^{\varepsilon -}_{app}|_{x=0} = 0 \\
\partial_x U^{\varepsilon +}_{app}|_{x=0} - \partial_x U^{\varepsilon -}_{app}|_{x=0} = 0 \\
U^{\varepsilon +}_{app}|_{x=\pi} = 0. \\
U^{\varepsilon \pm}_{app}|_{t=0}(x) = sin(x), \quad \{\pm x > 0\}, \\
\partial_t U^{\varepsilon \pm}_{app}|_{t=0} = 0, \quad \{\pm x > 0\},
\end{cases}
$$

where $R^{\varepsilon +} \in L^2(]0,\pi[\times \mathbb{R}^+)$ and $R^{\varepsilon -} \in L^2(]-\infty, 0[\times \mathbb{R}^+)$.

Let us assume that U_j^{*-} has been computed. The profile U_j^+ is then defined as the unique solution of the following 1-D wave equation:

$$
\begin{cases}
\partial_{tt} U_j^+ - c^2 \partial_{xx} U_j^+ = 0, \quad (x,t) \in]0,\pi[\times \mathbb{R}^+, \\
U_j^+|_{x=0} = U_j^{*-}|_{z=0}, \\
U_j^+|_{x=\pi} = 0, \\
U_j^+|_{t=0}(x) = 0, \quad \{x > 0\}, \\
\partial_t U_0^+|_{t=0} = 0, \quad \{x > 0\}.
\end{cases}
$$

We can thus compute the profile U^{*-}_{j+1} since it is the unique solution the following well-posed equation:

$$
\begin{cases}
U^{*-}_{j+1} - c^2 \partial_{zz} U^{*-}_{j+1} = -\partial_{tt} U_j^{*-}, \quad \{z < 0\}, \\
\partial_z U^{*-}_{j+1}|_{z=0} = \partial_x U_j^+|_{x=0}, \\
\lim\limits_{z \to -\infty} U^{*-}_{j+1} = 0.
\end{cases}
$$

Constructing the approximate solution at an order M large enough achieves the proof of Theorem 6.6.1.

6.6.2 Conclusion and perspectives

Let us answer the question asked in [PCLS05]: $U^{\varepsilon-}$ presents a boundary layer behavior in $\{x = 0^-\}$ since its approximate solution is composed **exclusively** of boundary layer profiles, which describes quick transitions at the boundary using a fast scale in ε. As a result of the loss in convergence induced by the boundary layers forming, we get the estimate stated in Theorem 6.6.1. In [PCLS05], the chosen small parameter is $\mu = \varepsilon^2$, hence, adopting the same notations as them, our estimate writes: $\|U^\mu - U\|_{L^\infty(]0,T[:L^2(]-\infty,\pi[)} = \mathcal{O}(\mu^{\frac{1}{4}})$, which is in agreement with the estimates given in [PCLS05]. Like in the penalization approach proposed by Rauch in [Rau79] and used by Bardos and Rauch in [BR82], as underlined by Droniou in [Dro97], boundary layers form on one side of the boundary. The approximation $U^{\varepsilon+}$ of U, is computed by taking $U^{\varepsilon+}|_{x=0} = U^{\varepsilon-}|_{x=0}$, thus, in numerical applications, the boundary layer phenomenon also affects the rate of convergence of $U^{\varepsilon+}$ towards U, as $\varepsilon \to 0^+$. In order to sharpen penalization methods used in numerical applications, an interesting question would be, in the same line of mind as in [FG07], to see whether there is some alternative method of penalization preventing or minimizing the formation of boundary layers.

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